Back to Maxwell's eqns w/o magnetic charge monopoles,

plug
$$\left.\begin{aligned}\vec{E} &= -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \nabla\times\vec{A}\end{aligned}\right]$$
② sol'n, into

$$① \left[\begin{aligned}\nabla\cdot E &= 4\pi\rho \\ \nabla\times B - \frac{1}{c}\frac{\partial E}{\partial t} &= \frac{4\pi}{c}\vec{J}\end{aligned}\right] \Rightarrow \begin{aligned} &-\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot\vec{A} = 4\pi\rho \\ &\nabla\times\nabla\times\vec{A} - \frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = \frac{4\pi}{c}\vec{J}\end{aligned}$$

use $\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$

& work in Lorentz gauge $\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} \stackrel{!}{=} 0$

get
$$\Box\phi = 4\pi\rho$$
$$\Box\vec{A} = 4\pi\vec{J}$$
$\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ "d'alembertian"

In empty space, $\rho = \vec{J} = 0$, can have travelling E.M. waves:

$$\phi = \phi_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)} \qquad \Box\phi = 0 \Rightarrow \omega^2 = c^2\vec{k}^2$$
$$\vec{A} = \vec{A}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)} \qquad \Box\vec{A} = 0 \Rightarrow \omega^2 = c^2\vec{k}^2$$

With $E = \hbar\omega$ $\vec{p} = \hbar\vec{k}$, this gives $E = c|\vec{p}|$

Massless particle dispersion relation

Gauge transform $\vec{A} \to \vec{A} + \nabla f \qquad \phi \to \phi - \frac{1}{c}\frac{\partial f}{\partial t}$

with $f = -i\alpha\, e^{i(\vec{k}\cdot\vec{x} - \omega t)} \qquad \alpha = \text{arbitrary}$

$$\Rightarrow \left.\begin{array}{l} \vec{A}_0 \to \vec{A}_0 + \alpha \vec{k} \\ \phi_0 \to \phi_0 - \frac{\alpha\omega}{c} \end{array}\right] \text{ Can use to take } \vec{A}_0 \perp \vec{k}$$

Note this doesn't affect the Lorentz gauge cond.

$\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} \Longrightarrow \vec{k}\cdot\vec{A}_0 - \frac{\omega}{c}\phi_0$ is unchanged

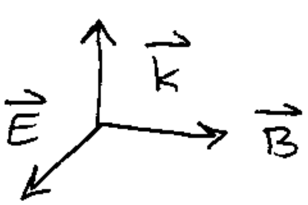
by above since $\omega = c|\vec{k}|$

So take $\vec{k}\cdot\vec{A} = \phi_0 = 0$

$\Rightarrow$ 2 polarizations possibilities for $\vec{A}_0$ ✓

$\vec{E} = i|\vec{k}|\,\vec{A}_0\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$ (take Re part)

$\vec{B} = i\vec{k}\times\vec{A}_0\, e^{i(\vec{k}\cdot\vec{x} - \omega t)} = \hat{k}\times\vec{E}$



Wave moves in $\vec{k}$ dir $\parallel$ to Poynting vector

$\vec{S} = \frac{c}{4\pi}\,\vec{E}\times\vec{B}$

$\vec{P}_{field} = \int d^3x\ \vec{S}/c^2$

More general sol'n to
With source included.

$$\begin{cases} \Box \phi = 4\pi\rho \\ \Box \vec{A} = 4\pi \vec{J} \end{cases}$$

travelling waves ⊕

$$\phi(\vec{x},t) = \int d^3\vec{x}' \; \frac{\rho\left(\vec{x}',\, t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}$$

$$\vec{A}(\vec{x},t) = \int d^3\vec{x}' \; \frac{\vec{J}\left(\vec{x}',\, t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}$$

Basic source example: point charges q_i, at positions $\vec{r}_i(t)$

$$\vec{\rho}(\vec{x},t) = \sum_i q_i \, \delta^3(\vec{x} - \vec{r}_i(t))$$

$$\vec{J}(\vec{x},t) = \sum_i q_i \, \vec{V}_i(t) \, \delta^3(\vec{x} - \vec{r}_i(t))$$

$$\vec{V}_i \equiv \frac{d}{dt}\vec{r}_i$$

Charge conservation $\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0$ ✓ Satisfied.

Part II: Relativity & Maxwell's eqns

Review relativity (Landau & Lifshitz vol 2)

We saw Maxwell's eqns $\rightarrow \Box \phi = 0$ & $\Box \vec{A} = 0$ & $\therefore$ $\Box \vec{E} = 0$ & $\Box \vec{B} = 0$ in vacuum. $\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$

$\Longrightarrow$ light has $\omega = c|\vec{k}|$ with velocity $c \simeq 2.998 \times 10^{10}$ cm/sec same for all observers. No specified reference frame, same speed of light for all.

Event 1: light emitted at $(t_1, \vec{X}_1)$
Event 2 light detected at $(t_2, \vec{X}_2)$ } coords in some ref frame.

Define $\Delta S \equiv c^2(t_2 - t_1)^2 - (\vec{X}_2 - \vec{X}_1)^2$

For light, velocity $= c \Longrightarrow \Delta S = 0$.

In another ' frame event 1 @ $(t_1', \vec{X}_1')$ & event 2 @ $(t_2', \vec{X}_2')$.

$$\Delta S' \equiv c^2(t_2' - t_1')^2 - (\vec{X}_2' - \vec{X}_1')^2$$

Since observer in ' frame also sees velocity $= c$
$\Delta S' = 0$ also.

Now argue for any 2 events

$ds^2 \equiv c^2dt^2 - d\vec{x}\cdot d\vec{x}$ is the same

in all reference frames. For light $ds^2 = 0$

& $ds'^2 = 0$, so generally can have

$ds^2 = a(|\vec{V}_{rel}|)\,ds'^2$

magnitude of relative velocity of moving frames

Consider ref frames 1,2,3

$ds_1^2 = a(|\vec{V}_{12}|)\,ds_2^2 = a(|\vec{V}_{13}|)\,ds_3^2$

$ds_2^2 = a(|\vec{V}_{23}|)\,ds_3^2$

$\therefore\ a(|\vec{V}_{12}|)\,a(|\vec{V}_{23}|) = a(|\vec{V}_{13}|)$

can only satisfy with $a = $ const. Take $\vec{V}_{rel} \to 0$

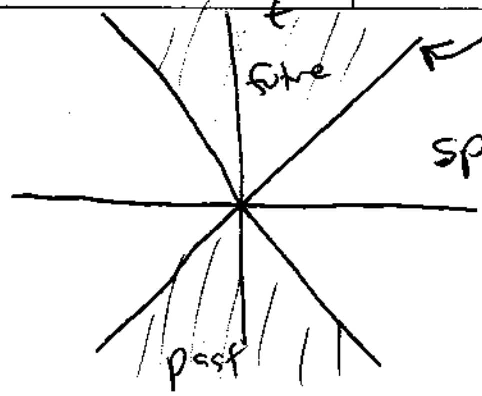
$\Rightarrow\ a \equiv 1$

$\therefore\ ds^2 \equiv c^2dt^2 - d\vec{x}\cdot d\vec{x}$ same in all ref frames.

$ds^2 = 0$: "light like separated events"

$ds^2 > 0$: "time like separated events, in causal contact

$ds^2 < 0$: "spacelike separated events" outside causal contact light cone: $|\vec{X}| = ct$



spacelike separated outside of causal contact

timelike separated events $\Delta S^2 > 0 \Rightarrow \exists$ a frame where events occur at same position $\Delta S^2 = \Delta t^2 - \Delta\vec{X}\cdot\Delta\vec{X} = \Delta t'^2 - \Delta\vec{X}'\cdot\Delta\vec{X}'$ with $\Delta\vec{X}' = 0$ in some frame.

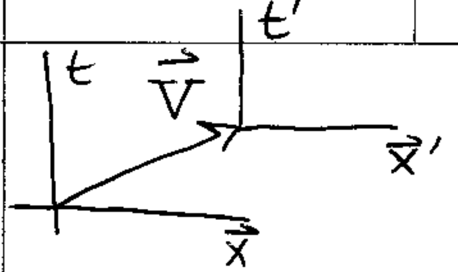
spacelike separated: $\Delta S^2 < 0 \Rightarrow \exists$ a frame where events are simultaneous, $\Delta t' = 0$.

Suppose we look at a moving clock. The time it reads is its propertime, with interval dt' & $d\vec{x}' = 0$. In our frame we see dt & $d\vec{x}$, with

$$ds^2 = c^2 dt^2 - |d\vec{x}|^2 = c^2 dt'^2$$

$$\therefore \; dt' = dt\sqrt{1 - \frac{1}{c^2}\left|\frac{d\vec{x}}{dt}\right|^2} = dt\sqrt{1 - \frac{|\vec{v}|^2}{c^2}}$$

also useful to write $dt' = \frac{1}{c} ds$



Pre relativity intuition: $\vec{x} = \vec{x}' + \vec{V}t$

$t = t'$

Relativity: $c^2 t^2 - \vec{x}^2 = c^2 t'^2 - \vec{x}'^2$

Take $\vec{V}$ along x-axis. Can see $y = y'$, $z = z'$

so $c^2 t^2 - x^2 = c^2 t'^2 - x'^2$

Hyperbolic rotation: $\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$

$(\cosh^2\psi - \sinh^2\psi = 1)$

Origin of ' frame @ $x' = 0$

$\Rightarrow ct = \cosh\psi \, ct'$, $x = \sinh\psi \, ct'$

$\therefore |V| \equiv v = \frac{x}{t} = c\frac{\sinh\psi}{\cosh\psi} = c\tanh\psi$

$\Rightarrow \cosh\psi = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \equiv \gamma \qquad \sinh\psi = \frac{V/c}{\sqrt{1-\frac{v^2}{c^2}}} \equiv \beta\gamma$

so $$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

inverse $$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

related by $V \to -V$ ✓

Aside 1: $y = y'$ & $z = z'$ by paint brushes on ends of meter stick argument given in lecture.

Another argument: 

view from behind

page: 

now boost: 

Same as start but $y \leftrightarrow y'$. So transf. must be $y \leftrightarrow y'$ symmetric. Only $y = y'$ satisfies this.

Aside 2

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

transpose

$$(ct \;\; x) = (ct' \;\; x') \begin{pmatrix} e & g \\ f & h \end{pmatrix}$$

$$\Rightarrow \; (ct \;\; -x) = (ct' \;\; -x') \begin{pmatrix} e & -g \\ -f & h \end{pmatrix}$$

$$ct^2 - x^2 = (ct \;\; -x) \begin{pmatrix} ct \\ x \end{pmatrix} = (ct' \;\; -x') \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

ensured if $\begin{pmatrix} e & -g \\ -f & h \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

take det

$\Rightarrow eh - fg = \pm 1 \;\; \Rightarrow$ choose $eh - fg = 1$

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix}^{-1} = \frac{1}{eh - fg} \begin{pmatrix} h & -f \\ -g & e \end{pmatrix} \overset{!}{=} \begin{pmatrix} e & -g \\ -f & h \end{pmatrix} \Rightarrow \boxed{h = e}, \; \boxed{f = g}$$

$$h^2 - g^2 = 1 \Rightarrow \boxed{\begin{matrix} h = \cosh\psi \\ g = \sinh\psi \end{matrix}}$$