Charge particles interact back & forth with electric-magnetic field. E&M field itself dynamical, contributes to action.

$$S = \int d^4x \ \mathcal{L} \qquad \text{with} \qquad \mathcal{L} = \mathcal{L}_m + \mathcal{L}_{int} + \mathcal{L}_{field}$$

$$\mathcal{L}_m = -c^2 D/\gamma \qquad \mathcal{L}_{int} = -\frac{1}{c} A_\mu J^\mu$$

$$\mathcal{L}_{field} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} = \frac{1}{8\pi}\left(\vec{E}^2 - \vec{B}^2\right)$$

$$= \frac{1}{8\pi}\left(\underbrace{\left(+\nabla\phi + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right)^2}_{\sim K.E.} - \underbrace{(\vec{\nabla}\times\vec{A})^2}_{\sim P.E.} \right)$$

Euler Lagrange eqns: $\displaystyle \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = \frac{\partial \mathcal{L}}{\partial A_\nu}$

$$\hookrightarrow \partial_\mu\left(-\frac{F^{\mu\nu}}{4\pi}\right) = -\frac{1}{c} J^\nu \implies \text{Maxwell eqns!} \checkmark$$

Note $\mathcal{L}_{field} = \frac{-1}{16\pi} F_{\mu\nu} F^{\mu\nu}$ is Lorentz invt.

2 basic Lorentz invts. from $F_{\mu\nu}$: $F_{\mu\nu} F^{\mu\nu} \sim \vec{E}^2 - \vec{B}^2$

& another: $\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \sim \vec{E} \cdot \vec{B}$.

Adding term $\sim \theta \, \vec{E} \cdot \vec{B}$ is naively trivial total derivative, but actually interesting "θ-term" Won't discuss further here. Nice argument of Landau & Lifshitz that these are only 2 indep. Lorentz invts.

$\vec{\mathcal{F}} \equiv \vec{E} + i\vec{B}$ Lorentz boost along x-axis

$$\begin{pmatrix} \mathcal{F}_x \\ \mathcal{F}_y \\ \mathcal{F}_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathcal{F}_x' \\ \mathcal{F}_y' \\ \mathcal{F}_z' \end{pmatrix} \qquad \alpha \equiv i \tanh^{-1}(v/c)$$

$\uparrow$ pure imaginary.

boosts = rotation by imag. angle. 3 indep. boosts + rotations $\longrightarrow$ 3 complex rotations. Only invt. of vector $\vec{\mathcal{F}}$ is its square $\vec{\mathcal{F}} \cdot \vec{\mathcal{F}} = (\vec{E}^2 - \vec{B}^2) + i\vec{E} \cdot \vec{B}$

$\hookrightarrow$ 2 indep. real invts.

$\mathcal{L}$ is Lorentz invt. ✓ & also gauge invt.

$\mathcal{L}_{field} = \frac{1}{8\pi}(\vec{E}^2 - \vec{B}^2)$ gauge invt. under $A_\mu \to A_\mu - \partial_\mu f$

$$\mathcal{L}_{int} = -\frac{1}{c} J_\mu A^\mu \longrightarrow \mathcal{L}_{int} + \frac{1}{c} J_\mu \partial^\mu f$$
$$= \mathcal{L}_{int} + \frac{1}{c} \partial^\mu (J_\mu f)$$

total div. integrates to bndy ~~term~~ can drop.

$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu$ is gauge inv.

under $A_\mu \to A_\mu - \partial_\mu f$. Note $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

is invariant, $\mathcal{L} \to \mathcal{L} + \frac{1}{c} J_\mu \partial^\mu f$

is invariant up to total derivative since in

$S = \int d^4x\, \mathcal{L}$ integrate by parts & use $\partial^\mu J_\mu = 0$.

Consider adding a term $\mathcal{L}_m = \frac{\mu^2}{2} A_\mu A^\mu$ to $\mathcal{L}$.

Would violate gauge invariance $\to$ no good

must take $\mu = 0$. The effect of this term

would give photon a mass, so gauge inv $\Leftrightarrow$

massless photon!

To see why $\mathcal{L}_m$ gives photon a mass, note new

Euler Lagrange eqns $\longrightarrow$ $\Box A^\nu + \mu^2 A^\nu = \frac{4\pi}{c} J^\nu$

in vacuum $J^\nu = 0$, $A^\nu = A_0^\nu e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$(\Box + \mu^2) A^\nu = 0 \implies \omega^2 = c^2 \vec{k}^2 + \mu^2 c^2$

$E = \hbar\omega$
$\vec{p} = \hbar\vec{k}$

$\implies E^2 = c^2 \vec{p}^{\,2} + m_\gamma^2 c^4$ $\qquad m_\gamma = \mu\hbar/c$

point charge sol'n is $\phi = \frac{q e^{-\mu r}}{r}$ "Yukawa potential"

Can show $\mu = 0$ to high accuracy by looking @ earth's $\vec{B}$ field.

Symmetries $\longleftrightarrow$ conservation laws

If $\mathcal{L}$ does not explicitly depend on x^μ
(only implicitly, via fields)

$\rightarrow$ x^μ translation invariace

$\Rightarrow$ conserved charges $P^\mu = (\underset{\uparrow}{\mathcal{H}}, \; c\vec{P})$

Hamiltonian

In anology w/ electric charge $Q = \int d^3x \, J^0$

Q consered $\longleftrightarrow$ $\partial_\mu J^\mu = 0$

$$P^\mu = \int d^3x \, T^{\mu 0} \quad \text{conserved}$$

$$\longleftrightarrow \quad \partial_\nu T^{\mu\nu} = 0$$

$T^{\mu\nu}$ = "stress-energy tensor" ~ 4 momentum current density

$$\frac{dP^\mu}{c\,dt} = \int_V d^3x \, \partial_0 T^{\mu 0} = -\int_V d^3x \, \partial_i T^{\mu i}$$

$$= -\oint_{\partial V} T^{\mu i} dS^i$$ = flux of 4-momentum out of surface of region

Consider collection of particles: $T^{\mu 0} = \sum_n c p_n^\mu \delta^3(\vec{x} - \vec{x}_n(t))$
(n labels particle)

$$T^{\mu\nu} = \sum_n p_n^\mu \frac{dx_n^\nu}{dt} \delta^3(\vec{x} - \vec{x}_n(t))$$

$$= \sum_n c^2 \frac{p_n^\mu p_n^\nu}{E_n} \delta^3(\vec{x} - \vec{x}_n(t))$$

note this is <u>symmetric</u> in $\mu \leftrightarrow \nu$, will be important.

Also $T^{\mu\nu} = \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} T^{\mu'\nu'}$ under Lorentz transformation since $p_n^\mu = \Lambda^\mu_{\mu'} p_n^{\mu'}$ and $\frac{1}{E_n}\delta^3(\vec{x} - \vec{x}_n(t))$ is Lorentz inv.

(d^4x Lorentz inv $\Rightarrow d^3x \sim \frac{1}{dx_0} \Rightarrow \delta^3(x) \sim dx^0$ and $E_n \sim dx^0$ under Lorentz transf so $\frac{1}{E_n}\delta^3(\vec{x})$ inv)

Check $\partial_\mu T^{\mu\nu} = 0$: $\quad \frac{\partial}{\partial x^i}$ equiv.

$$\frac{\partial}{\partial x^i} T^{i\nu} = \sum_n p_n^\nu \frac{dx_n^i}{dt} \left(-\frac{\partial}{\partial x_n^i}\right) \delta^3(\vec{x} - \vec{x}_n(t))$$

$$= \sum_n p_n^\nu \left(-\frac{\partial}{\partial t}\right) \delta^3(\vec{x} - \vec{x}_n(t))$$

$$= -\frac{\partial}{c\,\partial t}\left(T^{0\nu}\right) + \sum_n \left[\frac{\partial}{\partial t} p_n^\nu(t)\right] \delta^3(\vec{x} - \vec{x}_n(t))$$

$$\rightarrow \partial_\mu T^{\mu\nu} = \sum_n \frac{\partial p_n^\nu}{\partial t} \delta^3(\vec{x} - \vec{X}_n(t))$$

= density of external force, vanishes if $\mathcal{L}$ is translationally inv't. (more details soon)

Note $\partial_\mu T^{\mu\nu} = 0 \longrightarrow \frac{d}{dt} \underline{P}^\mu = -c \oint_{\partial V} T^{\mu i} dS^i$

$$\underline{P}^\mu = \int_V d^3x \, T^{\mu 0} = (E, c\underline{P})$$

so $\frac{dE}{dt} = -c^2 \oint_{\partial V} \frac{T^{0i}}{c} dS^i$ = energy flux out of surface

momentum density

$$E\vec{v} = m\vec{v}\gamma c^2 = c^2 \vec{p}$$

$$\frac{d\vec{p}^j}{dt} = - \oint \underbrace{T^{ij}} \, dS^i$$

area element

force on volume element

inward!

pressure & shear stresses

For free particles mass density: $D \equiv \sum_n m_n \delta^3(\vec{x} - \vec{x}_n(t))$

$$T^{\mu\nu}_{matter} = \frac{\varepsilon}{c^2} u^\mu u^\nu$$

ε = energy density = energy/vol.

$$\frac{\varepsilon}{c^2} = \frac{D}{\gamma}$$