9/28 Lecture outline

$\star$ **Reading for today's lecture: Luke, chapters 1 and 2; Tong up to section 2.7; Sredniki chapters 1-3.**

• Relativistic QM? Issue: particle number can change; can't have a relativistic single particle QM. Can't define $\vec{X}_{op}$ or $|\vec{x}\rangle$. In QM, observables, e.g. L_z, aren't attached to their location, so can have problems with causality. Want something like $[O_1(x_1), O_2(x_2)] = 0$ for spacelike separations, $(x_1 - x_2)^2 < 0$.

• Solution: quantum field theory. Replace particles, e.g. electrons, with fluctuations of a local field. Agrees with the fact that all electrons are the same. Whether here or on the other side of the universe, an electron is the same kind of blip of the electron field, which fills the universe.

• Conventions: $\hbar = c = 1$, mostly minus metric $g_{\mu\nu}$, e.g. $\partial_\mu = (\partial_t, \vec{\nabla})$, $\partial_\mu \partial^\mu \equiv \partial^2 = \partial_t^2 - \nabla^2$, $f(x) = \int \frac{d^n k}{(2\pi)^n} \widetilde{f}(k) e^{ikx}$, $\widetilde{f}(k) = \int d^n x f(x) e^{-ikx}$.

• "What goes wrong if we just do the S.E. with $H_{rel} = \sqrt{\vec{p}^2 + m^2}$?" Let's see. Start with $|\psi(t=0)\rangle = |\vec{x} = 0\rangle$. Compute

$$\begin{aligned} \langle \vec{x} | \psi(t) \rangle = \langle \vec{x} | e^{-iHt} | \vec{x} = 0 \rangle &= \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} e^{-i\sqrt{\vec{p}^2+m^2}t} \\ &= -\frac{i}{(2\pi)^2 r} \int_{-\infty}^{\infty} p dp e^{ipr} e^{-i\sqrt{p^2+m^2}t} \\ &= \frac{i e^{-mr}}{2\pi^2 r} \int_m^\infty dz z e^{-(z-m)r} \sinh(\sqrt{z^2 - m^2} t) \end{aligned}$$

The last step is by deforming the contour in the complex p plane, and getting contributions along the branch cut in the UHP, with $z = -ip$; the contribution along the big semi-circle at infinity vanishes for $r > t$. The integral is positive, so non-vanishing outside the forward light cone: acausal, with causality recovered as an approximation for $r \gg m$. In QFT, the difference will be antiparticles to the rescue! The antiparticle contribution is added, and cancels the acausality.

• Multiparticle states. E.g. $|\vec{k}_1, \vec{k}_2, \ldots \vec{k}_n\rangle$, with completeness

$$1 = |0\rangle\langle 0| + \sum_{n=1}^{\infty} \frac{1}{n!} \int \frac{d^3 \vec{k}_1}{(2\pi)^3} \cdots \frac{d^3 \vec{k}}{(2\pi)^3} |\vec{k}_1 \ldots \vec{k}_n\rangle\langle \vec{k}_1 \ldots \vec{k}_n|.$$

Introduce a box for the moment, to make momenta discrete. Can then count how many excitations of each momenta. Fock space description, like counting the excitation level of the SHO. Like, there, introduce creation and annihilation operators.

Recall $H = \frac{p^2}{2} + \frac{1}{2}\omega^2 x^2$ (setting $m = 1$, to avoid a later change in notation). Write $H = \omega(a^\dagger a + \frac{1}{2})$, where $[a, a^\dagger] = 1$. Then define $|0\rangle$ s.t. $a|0\rangle = 0$, and $|n\rangle = c_n (a^\dagger)^n |0\rangle$.

Likewise, define $a(\vec{k})$ and $a^\dagger(\vec{k})$ s.t. $[a(\vec{k}), a^\dagger(\vec{k}')] = \delta_{\vec{k}\vec{k}'}$ for momenta in a box (will generalize to continuous, with Dirac delta functions). The multiparticle states are then $\prod_{i=1}^n a^\dagger(\vec{k}_i)|0\rangle$, and $H = \sum_{\vec{k}} \omega(\vec{k}) a^\dagger(\vec{k}) a(\vec{k})$.

• Note that $d^4k\delta(k^2 - m^2)\theta(k_0) \to \frac{d^3k}{2\omega(k)}$ upon doing the k_0 integral. So normalize $\langle k'|k\rangle = (2\pi)^3 2\omega(k)\delta^3(\vec{k} - \vec{k}')$.

• Classical and quantum particle mechanics, $L(q_a, \dot{q}_a, t)$, $p_a = \partial L/\partial \dot{q}_a$, $\dot{p}_a = \partial L/\partial q_a$, $H = \sum_a p_a \dot{q}_a - L$. Get quantum theory by replacing Poisson brackets with commutators, $[q_a(t), p_b(t)] = i\delta_{ab}$. Recall $O_H(t) = e^{iHt} O_S e^{-iHt}$ and $i\frac{d}{dt}O_H(t) = [O_H(t), H]$.

• Classical field theory. E.g. scalars $\phi_a(t, \vec{x})$, with $S = \int d^4x \mathcal{L}(\phi_a, \partial_\mu \phi_a)$. Then $\Pi^\mu_a = \partial\mathcal{L}/\partial(\partial_\mu \phi_a)$, and E.L. eqns $\partial\mathcal{L}/\partial\phi_a = \partial_\mu \Pi^\mu_a$. Define $\Pi_a \equiv \Pi^0_a$. $H = \int d^3x(\Pi\dot{\phi}_a - \mathcal{L}) = \int d^3x \mathcal{H}$.

Example: $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2\phi^2)$. Get the Klein-Gordon equation. Can't interpret ϕ as a probability wavefunction because of solutions $E = \pm\sqrt{\vec{p}^2 + m^2}$. But we'll see that the KG equation is fine as a field theory. (The field has both creation and annihilation operators, corresponding to the $E = \pm\sqrt{\vec{k}^2 + m^2}$ solutions.) Treated it first as a classical field theory, and write the general solution by superposition. We'll quantize next.

Ended around here

Another example: $\mathcal{L} = \frac{i}{2}(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - \nabla\psi^* \cdot \nabla\psi - m\psi^*\psi$. Get EOM: $i\partial_t \psi = -\nabla^2\phi + m\psi$. Looks like S.E., but again don't want to interpret ψ as a probability amplitude – here it's a classical field.

• Quantum field theory: replace $q_a(t) \to \phi_a(t, \vec{x})$. QM is like QFT in zero spatial dimensions, with the field playing role of position before:

$$[\phi_a(\vec{x}, t), \Pi_b(\vec{y}, t)] = i\delta_{ab}\delta^3(\vec{x} - \vec{y}) \quad \textit{(Equal time commutators).}$$

Example: for the KG theory,

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3(2\omega(k))}[a(k)e^{-ikx} + a^\dagger(k)e^{ikx}].$$

Then get

$$[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3(2\omega)\delta^3(\vec{k} - \vec{k}').$$

$$H = \tfrac{1}{2}\int \frac{d^3k}{(2\pi)^2(2\omega)}\omega(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})).$$

Need to normal order the first term.

- Causality? Compute $[\phi(x), \phi(y)] = D(x-y) - D(y-x)$, where

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega(k)} e^{-ik(x-y)}.$$

Note that the commutator is a c-number, not an operator. For spacelike separation, $(x-y)^2 = -r^2$, get $D(x-y) = \frac{m}{2\pi^2 r} K_1(mr)$. Although $D(x-y) \sim \exp(-m|\vec{x}-\vec{y}|)$ is non-vanishing outside the forward light cone, the above difference is not. Good.

- Get more interesting theories by adding interactions, e.g. $V(\phi) = \frac{1}{2}m^2\phi^2 + \lambda\phi^4$, treat 2nd term as a perturbation.

- Symmetries of $\mathcal{L}$ and Noether's theorem. If a variation $\delta\phi_a$ changes $\delta L = \partial_\mu F^\mu$, then it's a symmetry of the action and there is a conserved current: $j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a - F^\mu$.

Example: $x^\mu \to x^\mu + \epsilon^\mu$, $\delta\phi_a = \epsilon^\nu \partial_\nu \phi_a$, $\delta\mathcal{L} = \epsilon^\nu \partial_\nu \mathcal{L}$ (assuming no explicit x dependence). Get $T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a}\partial_\nu\phi_a - g_{\mu\nu}\mathcal{L}$. Stress energy tensor. Conserved charge is $P_\mu = \int d^3\vec{x} T_{\mu 0}$.

Another example: $\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$, leads to conserved $M_{\mu\rho\sigma} = x_\mu T_{\rho\sigma} - x_\sigma T_{\rho\mu}$. Conserved charge is $M_{\rho\sigma} = \int d^3x M_{0\rho\sigma}$. Conserved angular momentum.

- Next time: Feynman propagator.