10/7 Lecture outline

$\star$ **Reading: Luke, chapters 3 and 4. Maybe a bit of chapter 5, if time.**

• Continue with green's functions. Last time: $\mathcal{L} = \frac{1}{2}\partial\phi^2 - \frac{1}{2}m^2\phi^2 - \rho\phi$, where ρ is a classical source. Solve by $\phi = \phi_0 + i\int d^4yD(x-y)\phi(y)$, where

$$(\partial_x^2 + m^2)D(x-y) = -i\delta^4(x-y),$$

which we can use to solve $(\partial_x^2 + m^2)\phi(x) = \rho(x)$, via $\phi(x) = \phi_0(x) + i\int d^4yD(x-y)\rho(y)$, where ϕ_0 is a solution of the homogeneous KG equation. Get

$$D_?(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}e^{-ik(x-y)}.$$

The ? is because we need to specify about how the poles are handled. Consider the k_0 integral in the complex plane. There are poles at $k_0 = \pm\omega_k$, where $\omega_k \equiv +\sqrt{\vec{k}^2+m^2}$. There are choices about whether the contour goes above or below the poles. Going above both poles gives the retarded green's function, $D_R(x-y)$ which vanishes for $x_0 < y_0$. Considering $x_0 > y_0$, get that

$$\begin{aligned} D_R(x-y) &= \theta(x_0-y_0)\int \frac{d^3k}{(2\pi)^3 2\omega_k}(e^{-ik(x-y)} - e^{ik(x-y)}) \\ &\equiv \theta(x_0-y_0)(D(x-y) - D(y-x)) = \theta(x_0-y_0)\langle[\phi(x),\phi(y)]\rangle, \end{aligned}$$

where

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}e^{-ik(x-y)}.$$

This is reasonable: then the $\rho(y)$ source only affects $\phi(x)$ in the future.

Going below both poles gives the advanced propagator, which vanishes for $y_0 < x_0$.

• Feynman propagator. Define

$$D_F(x-y) \equiv \langle T\phi(x)\phi(y)\rangle = \begin{cases} \langle\phi(x)\phi(y)\rangle & \text{if } x_0 > y_0 \\ \langle\phi(y)\phi(x)\rangle & \text{if } y_0 > x_0 \end{cases}.$$

Here T means to time order: order operators so that earliest is on the right, to latest on left. Object like $\langle T\phi(x_1)\ldots\phi(x_n)\rangle$ will play a central role in this class. Time ordering convention can be understood by considering time evolution in $\langle t_f|t_i\rangle$. Evaluate $D_F(x-y)$ by going to momentum space:

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik(x-y)},$$

where $\epsilon \to 0^+$ enforces that we go below the $-\omega_k$ pole and above the $+\omega_k$ pole, i.e. we get $D(x-y)$ if $x_0 > y_0$, and $D(y-x)$ if $x_0 < y_0$, as desired from the definition of time ordering. We'll see that this ensures causality.

• Define contraction of two fields $A(x)$ and $B(y)$ by $T(A(x)B(y)) - :A(x)B(y):$. This is a number, not an operator, e.g. for $x^0 > y^0$ the contraction is $[A^+, B^-]$, and for $y^0 > x^0$ it is $[B^+, A^-]$. So can put between vacuum states to get that the contraction is $\langle TA(x)B(y)\rangle$. For example, in the KG theory the contraction of $\phi(x)$ and $\phi(y)$ is $D_F(x-y)$.

• Simple example of interacting theory:

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi^2 - \mu^2\phi^2) + (\partial\psi^\dagger\partial\psi - m^2\psi^\dagger\psi) - g\phi\psi\psi^\dagger.$$

Toy model for interacting nucleons and mesons. Treat last term as a perturbation.

• Dyson's formula. Compute scattering S-matrices. Consider asymptotic in and out states, with the interaction turned off. Time evolve, with the interaction smoothly turned on and off in the middle.

$$|\psi(t)\rangle = Te^{-i\int d^4x\mathcal{H}_I}|i\rangle.$$

Motivate it by solving $i\frac{d}{dt}|\psi(t)\rangle = H_I(t)|\psi(t)\rangle$ iteratively:

$$|\psi(t)\rangle = |i\rangle + (-i)\int_{-\infty}^{t} dt_1 H_I(t_1)|\psi(t_1)\rangle$$

$$|\psi(t_1)\rangle = |i\rangle + (-i)\int_{-\infty}^{t_1} dt_1 H_I(t_2)|\psi(t_2)\rangle$$

etc where $t_1 > t_2$, and then symmetrize in t_1 and t_2 etc., which is what the T time ordering does.

Now use Wick's theorem:

$$T(\phi_1\dots\phi_n) =: \phi_1\dots\phi_n : + \sum_{contractions} : \phi_1\dots\phi_n :$$

to get rid of the time ordered products. Thereby compute probability amplitude for a given process

$$\langle f|(S-1)|i\rangle = i\mathcal{A}_{fi}(2\pi)^4\delta^{(4)}(p_f - p_i).$$

• Look at some examples, and connect with Feynman diagrams. As a first, simple example consider the above theory, with $H_{int} = \int d^3x g\phi\psi^\dagger\psi$. Use $\phi \sim a + a^\dagger$ for "mesons,"

$\psi \sim b + c^\dagger$, and $\psi^\dagger \sim b^\dagger + c$. We'll say that b annihilates a nucleon N and $c^\dagger$ creates an anti-nucleon $\bar{N}$. Conservation law, conserved charge $Q = N_b - N_c$.

Example: meson decay. $|i\rangle = a^\dagger(p)|0\rangle$, $|f\rangle = b^\dagger(q_1)c^\dagger(q_2)|0\rangle$. Compute $\langle f|S|i\rangle = -ig\delta^4(p - q_1 - q_2)$ to $\mathcal{O}(g)$.

Now consider $N + N \to N + N$, to $\mathcal{O}(g^2)$. The initial and final states are

$$|i\rangle = b^\dagger(p_1)b^\dagger(p_2)|0\rangle, \qquad \langle f| = \langle 0|b(p_1')b(p_2').$$

The term that contributes to scattering at $\mathcal{O}(g^2)$ is

$$T\frac{(-ig)^2}{2!}\int d^4x_1 d^4x_2 \phi(x_1)\psi^\dagger(x_1)\psi(x_1)\phi(x_2)\psi^\dagger(x_2)\psi(x_2).$$

The term that contributes to $S-1$ thus involves

$$\langle p_1'p_2'| : \psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2) : |p_1p_2\rangle = \langle p_1'p_2'| : \psi^\dagger(x_1)\psi^\dagger(x_2)|0\rangle\langle 0|\psi(x_1)\psi(x_2)|p_1, p_2\rangle.$$

$$= \left(e^{i(p_1'x_1 + p_2'x_2)} + e^{i(p_1'x_2 + p_2'x_1)}\right)\left(e^{-i(p_1x_1+p_2x_2)} + e^{-i(p_1x_2+p_2x_1)}\right).$$

The amplitude involves this times $D_F(x_1 - x_2)$ (from the contraction), with the prefactor and integrals as above. The final result is

$$i(-ig)^2\left[\frac{1}{(p_1 - p_1')^2 - \mu^2} + \frac{1}{(p_1 - p_2')^2 - \mu^2}\right](2\pi)^4\delta^{(4)}(p_1 + p_2 - p_1'p_2').$$

Explicitly, in the CM frame, $p_1 = (\sqrt{p^2 + m^2}, e\widehat{e})$ and $p_2 = (\sqrt{p^2+m^2}, -p\widehat{e})$, $p_1' = (\sqrt{p^2+m^2}, p\widehat{e'})$, $p_2' = (\sqrt{p^2+m^2}, -p\widehat{e'})$, where $\widehat{e}\cdot\widehat{e}' = \cos\theta$, and get

$$\mathcal{A} = g^2\left(\frac{1}{2p^2(1-\cos\theta) + \mu^2} + \frac{1}{2p^2(1+\cos\theta)+\mu^2}\right).$$

The scattering by ϕ exchange leads to an attractive Yukawa potential. Indeed, the first term in the above amplitude gives, upon using $(p_1 - p_1')^2 - \mu^2 = |\vec{p}_1 + \vec{p}_1'|^2 + \mu^2$, and the Born approximation in NRQM, $\mathcal{A}_{NR} = -i\int d^3\vec{r}e^{-i(\vec{p}'-\vec{p})}U(\vec{r})$, the attractive Yukawa potential

$$U(r) = \int \frac{d^3p}{(2\pi)^3}\frac{-g^2e^{i\vec{q}\cdot\vec{r}}}{|\vec{q}|^2 + \mu^2} = -\frac{g^2}{4\pi r}e^{-\mu r}.$$

This gives Yukawa's explanation of the attraction between nucleons, from meson exchange.

• Feynman diagrams. Each vertex gets $(-ig)(2\pi)^4\delta^4(p_{total\ in})$, each internal line gets $\int\frac{d^4k}{(2\pi)^4}D_F(k^2)$. Result is $\langle f|(S-1)|i\rangle$.