10/24 Lecture outline

• EL equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0.$$

We showed that these equations imply that

$$\frac{d}{dt}[\sum_i q_i \frac{\partial L}{\partial \dot{q}_i} - L] = -\frac{\partial L}{\partial t},$$

so if L doesn't depend explicitly on t, then

$$\sum_i q_i \frac{\partial L}{\partial \dot{q}_i} - L = constant (= E).$$

• 'Cyclic" coordinates ($\partial L/\partial q_{cyclic} = 0$) and $p_{cyclic} = constant$ conservation law.

• Last time, example of sliding point mass on sliding wedge. $L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X} + \dot{x})^2 + \frac{1}{2}m\dot{x}^2 \tan^2\alpha - mgx\tan\alpha$. Here X is a cyclic coordinate and conserved quantity p_X is the expected momentum conservation.

• Example of motion in 2d central potential $U = U(r)$,

$$\mathcal{L} = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

so ϕ is a cyclic coordinate, $\partial L/\partial\phi = 0$, and correspondingly $p_\phi = mr^2\dot{\phi} = \ell$ is conserved. This is related to the rotation symmetry, $\phi \to \phi +$ constant, as we'll soon discuss. The r equation of motion (EOM) is

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} = \frac{\partial\mathcal{L}}{\partial r} \to m\frac{d^2}{dt^2}r = mr\dot{\phi}^2 - U'(r).$$

We can now eliminate $\dot{\phi}$ in favor of ℓ to get

$$m\frac{d^2r}{dt^2} = \frac{\ell^2}{mr^3} - U'(r) \equiv -\frac{d}{dr}U_{eff}, \qquad U_{eff} = U(r) + \frac{\ell^2}{2mr^2}.$$

Here U_{eff} is an effective potential which accounts for the centrifugal force of the rotating object. The energy is also conserved:

$$H = p_r\dot{r} + p_\phi\dot{\phi} - \mathcal{L} = \frac{p_r^2}{2m} + U_{eff}(r).$$

We can think of this as an effective 1d problem with

$$\mathcal{L}_{eff} = \tfrac{1}{2}m\dot{r}^2 - U_{eff}(r).$$

Caution: it is important that we eliminated $\dot\phi$ in favor of ℓ only **after** computing the Euler-Lagrange equations of motion for r. If we had replaced $mr^2\dot\phi \to \ell$ directly in the original Lagrangian we would have obtained **not** the above $\mathcal{L}_{eff}$ but instead one where the $\ell^2/2mr^2$ term has the wrong sign. The mistake is because the E.L. equation for r has partial derivatives like $\frac{\partial}{\partial r}$ where ϕ and $\dot\phi$ are supposed to be held constant. On the other hand, if we plug ℓ back into $\mathcal{L}$ directly, we make the mistake of instead holding $\ell = mr^2\dot\phi$ constant. The particle has 2 degrees of freedom, and it is important to compute the EOM for the two independent coordinates, before using the conservation law to eliminate the cyclic coordinate.

• Noether's theorem: continuous symmetries implies conservation laws. If $\mathcal{L}$ is invariant under $q_i \to q_i(\xi)$, with infinitesimal change $\delta q_i = \frac{\partial q_i}{\partial \xi}\delta\xi$, then

$$0 = \delta L = \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot q_i}\delta \dot q_i = \frac{d}{dt}(\frac{\partial L}{\partial \dot q_i}\delta q_i),$$

from which it follows that

$$\frac{\partial L}{\partial \dot q_i}\frac{\partial q_i(\xi)}{\partial \xi}|_{\xi=0}$$

is a conserved quantity.

• If L is independent of a coordinate q, then there is a symmetry $q \to q + \xi$, and the conserved quantity is simply the corresponding conjugate momentum $p = \partial L/\partial \dot q$. The above result is more non-trivial when the symmetry is less obvious. Example: particle in 2d with $U = U(r)$ has symmetry $\phi \to \phi + \xi$, gives conserved $\ell = mr^2\dot\phi$. (Or use above in rectangular coordinates.)

• Example of helical symmetry: $U(\rho, \phi, z) = U(\rho, a\phi + z)$ has symmetry $\phi \to \phi + \xi$, $z \to z - \xi a$, gives conserved quantity $m\rho^2\dot\phi - ma\dot z$.

• System of particles has translation symmetry $\vec x_a \to \vec x_a + \xi\vec n$, gives conserved quantity $\vec n \cdot \vec P$, where $\vec P$ is the total momentum. Conservation of momentum.

• Rotation symmetry: $\delta \vec x_a = \xi \widehat n \times \vec x_a$ gives conserved quantity

$$\sum_a \frac{\partial L}{\partial \dot{\vec x}_a} \cdot \widehat n \times \vec x_a = \widehat n \cdot \vec L, \qquad \vec L = \sum_a \vec x_a \times \vec p_a,$$

i.e. conservation of angular momentum.

• Now consider time translations $t \to t + \delta t$. When L does not depend explicitly on t, this is a symmetry of the action, and the corresponding conserved quantity is the Hamiltonian

$$H = \sum_i \frac{\partial L}{\partial \dot q_i}\dot q_i - L.$$

When this is conserved, it is the energy.

• $H = H(q, p, t)$. Find dH and show $\dot{q} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial q$ and $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$.

• Get $H = T + U$ if Cartesian $\vec{r}_a = \vec{r}_a(q_i)$ is t independent. Example of bead on spinning hoop. $T = \frac{1}{2}ma^2(\dot{\theta}^2 + \omega^2 \sin^2\theta)$, $U = mga(1 - \cos\theta)$.

• Example of charged particle in electric and magnetic fields, $\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$, $\vec{B} = \nabla \times \vec{A}$.

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - q\phi + q\dot{\vec{r}} \cdot \vec{A}.$$

Get $\vec{p} = m\vec{v} + q\vec{A}$. Gauge invariance. Hamiltonian.