10/28 Lecture outline

• Last time, conservation of momentum and angular momentum from spatial translation and rotation symmetry, respectively. Now consider time translations $t \to t + \delta t$. Recall that we showed before that

$$\frac{d}{dt}[\sum_i q_i \frac{\partial L}{\partial \dot{q}_i} - L] = -\frac{\partial L}{\partial t}.$$

When L does not depend explicitly on t, time translations is a symmetry of the action, and the corresponding conserved quantity is the Hamiltonian

$$H = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L.$$

When this is conserved, it is the energy.

• $H = H(q, p, t)$. Find dH and show $\dot{q} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial q$ and $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$.

• Get $H = T + U$ if Cartesian $\vec{r}_a = \vec{r}_a(q_i)$ is t independent.

• Example where H is conserved but $H \neq T + U$, bead on spinning hoop. $T = \frac{1}{2}ma^2(\dot{\theta}^2 + \omega^2 \sin^2\theta)$, $U = -mga\cos\theta$. Since $\frac{\partial L}{\partial t} = 0$, we have the conserved quantity

$$H_{bead} = p_\theta \dot{\theta} - L = \tfrac{1}{2}ma^2\dot{\theta}^2 - \tfrac{1}{2}ma^2\omega^2\sin^2\theta - mga\cos\theta,$$

which differs from $E_{bead} = T + U$, $H_{bead} = E_{bead} - ma^2\omega^2\sin^2\theta(t)$. Indeed, E_{bead} is not constant, because external driver that's spinning the hoop is doing t dependent work W_{ext} on the system.

• Example of charged particle in electric and magnetic fields, $\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$, $\vec{B} = \nabla \times \vec{A}$.

$$L = \tfrac{1}{2}m\dot{\vec{r}}^2 - q\phi + q\dot{\vec{r}} \cdot \vec{A}.$$

Get $\vec{p} = m\vec{v} + q\vec{A}$. Gauge invariance. Hamiltonian.

• Now discuss systems with constraints. Example: pendulum:

$$\mathcal{L} = \tfrac{1}{2}m\ell^2\dot{\phi}^2 + mg\ell\cos\phi$$

has 1 d.o.f., namely ϕ. Alternatively, we could use $x_{bob} = \ell\sin\phi$ and $y_{bob} = -\ell\cos\phi$, with the constraint $x_{bob}^2 + y_{bob}^2 = \ell^2$. This is an example of a *holonomic constraint*, which more generally are constraints of the form:

$$f(q_i, t) = 0.$$

We want to extremize the action, subject to the requirement that the variations δq_i should satisfy the constraint. A way to do this is to introduce a Lagrange multiplier. We replace the Lagrangian with

$$\mathcal{L} + \lambda f,$$

where λ is the Lagrange multiplier.