11/7 Lecture outline

• Last time: Two body central force problems, reduces to solving the 1d problem:

$$\mu \frac{d^2 r}{dt^2} = -\frac{dU_{eff}(r)}{dr}, \qquad U_{eff} = U(r) + \frac{\ell^2}{2\mu r^2}.$$

• Conservation of energy (since $U(r)$ is time translational invariant). In the CM frame,

$$H = \tfrac{1}{2}\mu \dot{r}^2 + U_{eff}(r) = E = \text{constant}.$$

• Example of $U_{eff}(r) = -\frac{Gm_1 m_2}{r} + \frac{\ell^2}{2\mu r^2}$. Illustrate turning points r_{min} and r_{max} for case $E < 0$: bounded orbit. For $E > 0$, there is a r_{min} but no r_{max}: unbounded orbit.

• For bounded orbits, we have from the virial theorem (using that $U \sim q^{k=-1}$), $\langle U \rangle = 2E$, $\langle T \rangle = -E$.

• Using above equations, we can solve the problem, reducing it to the computation of two integrals. Rewrite the energy conservation equation as

$$dt = \frac{dr}{\sqrt{\frac{2}{m}(E - U(r) - \frac{\ell^2}{2mr^2}}}$$

and integrate to get

$$t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}(E - U(r) - \frac{\ell^2}{2\mu r^2}}},$$

which can be inverted to find $r(t)$. Then rewrite the conservation of angular momentum equation as

$$d\phi = \frac{\ell dt}{\mu r^2}$$

and integrate both sides to get

$$\phi - \phi_0 = \ell \int_0^t \frac{dt}{\mu r^2(t)}.$$

We thus have obtained, in principle, $r(t)$ and $\phi(t)$.

• Discuss nearly circular orbits. Circular orbit at points $r = r_0$ where $U'_{eff}(r_0) = 0$. Stable if $U''(r_0) > 0$. Then consider nearly circular orbits by expanding $r = r_0 + \epsilon(t)$ and find

$$\frac{d^2\epsilon}{dt^2} = -\frac{U''_{eff}(r_0)}{\mu}\epsilon \equiv -\omega^2 \epsilon,$$

which has solution $\epsilon = \epsilon_0 \cos \omega t$, with $\omega \sqrt{U''_{eff}(r_0)/\mu}$ the frequency of oscillation about $r = r_0$.