10/5 Lecture outline

$\star$ **Reading for today's lecture: Taylor chapters 3, and 4. Also consult Arovas notes.**

• Last time, for conservative forces, $\vec{F}(\vec{r},t) = -\nabla U(\vec{r},t)$, and then

$$dU = \nabla U \cdot d\vec{r} + \frac{\partial U}{\partial t} dt = -\vec{F} \cdot d\vec{r} + \frac{\partial U}{\partial t} dt,$$

from which it follows that $d(T+U) = \frac{\partial U}{\partial t} dt$, so *mechanical* energy $E = T+U$ is conserved if U is time independent, $\partial U/\partial t = 0$. (Total energy conserved in any case, but mechanical energy can be converted into other types of energy).

• Plot $U(x)$, stable and unstable equilibrium positions, turning points, etc.

• Example from book: cube of side length $2b$, and mass m, resting on cylinder of radius r, with point of contact at an angle θ from the horizontal. It has $U = mgh = mg(r+b)\cos\theta + mgr\theta\sin\theta$. The second term is thanks to a "no-slip condition" (infinite friction), meaning that the cube has to rotate as the point of contact, labeled by θ varies. Find $\theta = 0$ is a point of stable equilibrium if $b < r$ and unstable if $b > r$: $d^2U/d\theta^2 = mg(r-b)$. Expanding around $\theta = 0$ get $U(\theta) = U_0 + \frac{1}{2}mg(r-b)\theta^2 + \mathcal{O}(\theta^3)$, so there is a restoring force and stable equilibrium at $\theta = 0$ if $r < b$. Oscillates with $\omega = \sqrt{g(r-b)}$.

• Using energy conservation to solve the EOM in 1d: $\frac{1}{2}m\dot{x}^2 + U(x) = E$ gives

$$\int_{t_0}^{t} dt = t - t_0 = \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dx}{\sqrt{E - U(x)}},$$

which is inverted to find the solution $x(t)$.

• Energy of interaction of two particles, e.g. gravitational attraction between mass m_1 at $\vec{r}_1$ and mass m_2 and $\vec{r}_2$ is given by $U(\vec{r}) = -Gm_1m_2/|\vec{r}|$, where $\vec{r} = \vec{r}_1 - \vec{r}_2$. Translation invariance implies that it only depends on $\vec{r}$, and it follows from this that $\vec{F}_{12} = -\vec{F}_{21}$, and hence that momentum is conserved. The energy is $E = T_1 + T_2 + U(\vec{r})$, and is conserved.

• With many particles, we have $T = \sum_i T_i$ and $U = \sum_i (U_i^{ext} + \sum_{j>i} U_{ij})$.

• Example of harmonic oscillator: $U(x) = \frac{1}{2}m\omega^2x^2$, and $E = \frac{1}{2}m\omega^2A^2$ gives

$$\omega(t - t_0) = \int_{x_0/A}^{x/A} \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(x/A) - \sin^{-1}(x_0/A),$$

so $x(t) = A\sin(\omega t + \phi)$, where the phase shift ϕ is given by solving $x_0 = A\sin(\omega t_0 + \phi)$.

- Expand 1d potential $U(x)$ around critical point x_*: $U(x) = U(x_c) + \frac{1}{2}U''(x_*)(x - x_*)^2 + \mathcal{O}(x - x_*)^3$. For small deviations away from a point of stable equilibrium, always approximately a SHO, with spring constant $k = U''(x_0)$. This is why we like harmonic oscillators so much, they can be readily analyzed, and the results apply any time there are small oscillations around a stable equilibrium - this occurs all over the place in Nature.
- Next time: Damped harmonic oscillator

$$\left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2\right) x = 0,$$

(with $\gamma = 2\beta m$).