11/7 Lecture outline

⋆ **Reading for today's lecture: Luke chapter VIII**

• Finish up from last time: we saw

$$d\Gamma_{1\to 2} = \frac{|\mathcal{A}|^2 D_{2-body}}{2M}$$

$$d\sigma_{2\to 2} = \frac{|\mathcal{A}|^2}{4E_1E_2} D_{2-body} \frac{1}{|\vec{v}_1 - \vec{v}_2|}$$

$$D_{2-body(CM)} = \frac{p_1 d\Omega_1}{16\pi^2 E_{CM}} \qquad \text{(divide by 2! if identical final states).}$$

Examples: For $\mu^2 > 4m^2$, consider $\phi \to \bar{N}N$ decay. $\mathcal{A} = -g$, and get

$$\Gamma = \frac{g^2}{2\mu} \frac{p_1}{16\pi^2\mu} \int d\Omega_1 = \frac{g^2}{8\pi\mu^2} \frac{\sqrt{\mu^2 - 4m^2}}{2},$$

where the last factor is p_1. For $2 \to 2$ scattering in the CM frame,

$$d\sigma = \frac{|\mathcal{A}|^2}{4E_1E_2} \frac{p_f d\Omega_1}{16\pi^2 E_T} \frac{1}{|\vec{v}_1 - \vec{v}_2|} = \frac{|\mathcal{A}|^2 p_f d\Omega_1}{64\pi^2 p_i E_T^2}$$

where we used $|\vec{v}_1 - \vec{v}_2| = p_1(E_1^{-1} + E_2^{-2}) = p_i E_T / E_1 E_2$ in the CM frame, and p_i is the magnitude of the initial momentum, and p_f is that of the final momentum.

• Could discuss some interesting things here with the optical theorem etc, but won't. Assume it'll be covered next quarter.

• Next topic (Luke ch. VIII):

• Consider full interacting theory, with Hamiltonian H. Define the true vacuum $|\Omega\rangle$ such that $H|\Omega\rangle = 0$, and $\langle\Omega|\Omega\rangle = 1$. The true vacuum of an interacting QFT is a complicated beast – it can be thought of roughly as a soup of particle-antiparticle states – it can not be solved for solved for exactly, but that's generally OK. (Progress: in classical mechanics, can solve 2 body problem exactly, but ≥ 3 body only approximately; in GR, can solve 1 body problem exactly, but ≥ 2 body only approximately; in QM can generally solve even only 1-body problem only approximately, but at least the 0-body problem is trivial; in QFT, even the 0-body problem is not exactly solvable.)

Define Green functions or correlation functions by

$$G^{(n)}(x_1, \ldots x_n) = \langle\Omega|T\phi_H(x_1)\ldots\phi_H(x_n)|\Omega\rangle,$$

where $\phi_H(x)$ are the full Heisenberg picture fields, using the full Hamiltonian.

Now show that

$$G^{(n)}(x_1 \dots x_n) = \frac{\langle 0|T\phi_{1I}(x_1)\dots\phi_{nI}(x_n)S|0\rangle}{\langle 0|S|0\rangle},$$

where $|0\rangle$ is the vacuum of the free theory, and ϕ_{iI} are interaction picture fields, and the S in the numerator and denominator gives the interaction-Hamiltonian time evolution from $-\infty$ to x_n, then from x_n to x_{n-1} etc and finally to $t = +\infty$. To show it, take $t_1 > t_2 \dots > t_n$ and put in factors of $U_I(t_a, t_b) = T\exp(-i\int_{t_a}^{t_b} H_I)$ to convert ϕ_I to ϕ_H, using $\phi_H(x_i) = U_I(t_i, 0)^\dagger \phi_I(x_i) U_I(t_i, 0)$. Get $\langle 0|U_I(\infty, t_1)\phi_H(t_1)\dots\phi_H(t_n)U_I(t_n, -\infty)|0\rangle$, and U_I at ends can be replaced with full $U(t_1, t_2)$, since $H_0|0\rangle = 0$ anyway. Now use

$$\begin{aligned}
\langle\Psi|U(t,-\infty)|0\rangle &= \langle\Psi|U(t,-\infty)\left(|\Omega\rangle\langle\Omega| + \sum\int |n\rangle\langle n|\right)|0\rangle \\
&= \langle\Psi|\Omega\rangle\langle\Omega|0\rangle + \lim_{t'\to-\infty}\sum\int e^{iE_n(t'-t)}\langle\Psi|n\rangle\langle n|0\rangle \\
&= \langle\Psi|\Omega\rangle\langle\Omega|0\rangle
\end{aligned}$$

where 1 was inserted as a complete set of states, including the vacuum and single and multiparticle states, including integrating over their momenta, but the wildly oscillating factor kills all those terms. (Riemann-Lebesgue lemma: $\lim_{t\to\infty}\int d\omega f(\omega)e^{i\omega t} = 0$ for nice $f(\omega)$) The result follows upon doing the same for the denominator.

The $\langle 0|S|0\rangle$ in the denominator eliminates the vacuum bubble diagrams. So we have

$$G^{(n)}(x_1, \dots x_n) = \sum \text{Feynman graphs without vacuum bubbles.}$$

• Example: $G^{(4)}(x_1, x_2, x_3, x_4)$ in $\lambda\phi^4/4!$ theory. For each line from x to y, get a factor of $\Delta_F(x-y)$, and for each vertex at y get $-i\lambda\int d^4y$. Includes connected and disconnected diagrams. Disconnected ones will go away when computing S-matrix elements.

• It's more convenient often to work in momentum space,

$$\widetilde{G}^{(n)}(p_1, \dots p_n) = \int\prod_{i=1}^{n} d^4x_i e^{-ip_ix_i} G^{(n)}(x_1 \dots x_n).$$

Similar to what we computed before to get S-matrix elements, but the external legs include their propagators, and the external momenta are not on-shell.

• From Green functions $\widetilde{G}^{(n)}(p_1, \ldots, p_n)$, computed with external leg propagators, allowed to be off-shell, to S-matrix elements. E.g.

$$\langle k_3, k_4|S-1|k_1 k_2\rangle = \prod_{n=1}^{4} \frac{k_n^2 - m_n^2}{i\sqrt{Z}} \widetilde{G}(-k_3, -k_4, k_1, k_2),$$

where the factors are to amputate the external legs. Consider for example $\widetilde{G}^{(4)}(k_1, k_2, k_3, k_4)$ for 4 external mesons in our meson-nucleon toy model. The lowest order contribution is at $\mathcal{O}(g^0)$ and is

$$(2\pi)^4 \delta^{(4)}(k_1 + k_4) \frac{i}{k_1^2 - \mu^2 + i\epsilon} (2\pi)^4 \delta^{(4)}(k_2 + k_3) \frac{i}{k_2^2 - \mu^2 + i\epsilon} + 2 \text{ permutations.}$$

This is the -1 that we subtract in $S-1$, and indeed would not contribute to $2 \to 2$ scattering using the above formula, because it is set to zero by $\prod_{n=1}^{4}(k_n^2 - m_n^2)$ when the external momenta are put on shell. To get a non-zero result, need a $\widetilde{G}^{(4)}$ contribution with 4 external propagators, which we get e.g. at $\mathcal{O}(g^4)$ with an internal nucleon loop.

• Account for bare vs full interacting fields. Let $|k\rangle$ be the physical one-meson state of the full interacting theory, normalized to $\langle k'|k\rangle = (2\pi)^3 2\omega_k \delta^{(3)}(\vec{k}' - \vec{k})$. Then

$$\langle k|\phi(x)|\Omega\rangle = \langle k|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|\Omega\rangle = e^{ik\cdot x}\langle k|\phi(0)|\Omega\rangle \equiv e^{ik\cdot x} Z_\phi^{1/2}.$$

Can rescale the fields, s.t. $\langle k|\phi_R(x)|\Omega\rangle = e^{-ik\cdot x}$. The LSZ formula is:

$$\langle q_1 \ldots q_n|S-1|k_1 \ldots k_m\rangle = \prod_{a=1}^{n} \frac{q_a^2 - m_a^2}{i\sqrt{Z}} \prod_{b=1}^{m} \frac{k_b^2 - m_b^2}{i\sqrt{Z}} \widetilde{G}^{(n+m)}(-q_1, \ldots - q_n, k_1, \ldots k_m),$$

where the Green function is for the Heisenberg fields in the full interacting vacuum.