11/28 Lecture outline

⋆ **Reading: Luke chapter 10. Tong chapter 5**

• Today we'll compute amplitudes in the (spin $\frac{1}{2}$) nucleon + (scalar) meson toy model. Applications: this is Yukawa's original model for explaining the attraction between nucleons. It works. We'll see how the potential is always attractive, whether the nucleon charges are the same or opposite sign. This model will also set the stage for quantum electrodynamics (QED), where the scalar meson is replaced with the spin 1 photon and the nucleons are replaced with electrons. Here the rule that opposites attract and same sign charges repel comes from the difference between spin 1 vs spin 0 force carries. Finally, this model illustrates how the Higgs scalar interacts with the fundamental fermions of Nature.

Tinkertoy pieces:

$$\mathcal{L} \supset \bar{\psi}(i\partial\!\!\!/ - m)\psi \qquad \to \qquad \text{fermion propagator:} \qquad \frac{i}{p\!\!\!/ - m + i\epsilon},$$

$$\mathcal{L} \supset \tfrac{1}{2}\partial\phi\partial\phi - \tfrac{1}{2}\mu^2\phi^2 \qquad \to \qquad \text{scalar propagator:} \qquad \frac{i}{p^2 - \mu^2 + i\epsilon},$$

$$\mathcal{L} \supset -g\phi\bar{\psi}_a\Gamma_{ab}\psi_b(x) \qquad \to \qquad \text{scalar, fermion vertex} \qquad -ig\Gamma,$$

where the index a, b runs over the four fermion components (spin up and down for fermion and anti-fermion), so Γ is a 4×4 matrix (natural choices are $\Gamma = 1_{4\times 4}$ or $\Gamma = i\gamma_5$, where recall $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, and the i is there to keep $\mathcal{L}^\dagger = \mathcal{L}$, since $(\gamma^0\gamma_5)$ is anti-hermitian).

Incoming fermions get a factor of $u^r(p)$, outgoing fermions get $\bar{u}^r(p)$; incoming antifermions gets $\bar{v}^r(p)$, and outgoing antifermions get $v^r(p)$. The amplitude has indices $r = 1, 2$ for each external fermion, which accounts for the external fermion's spin. For internal fermion propagators we sum over the four fermion indices, which is accomplished by matrix multiplication of the above tinkertoy pieces, with Tr put in as appropriate. Write the amplitude by following the arrows backwards, from the head to the tail.

• Minus sign of fermion loop. This follows from working through the Dyson/Wick procedure, accounting for the minus signs when fermions are exchanged, as needed to bring contracted fermions next to each other. This relative minus sign for fermion vs boson loops plays a big role in supersymmetry.

• Examples of amplitudes, computed to lowest non-trivial order:

$N + \phi \to N + \phi$:

$$i\mathcal{A} = (-ig)^2\bar{u}^{r'}(p')\Gamma\left(\frac{i(p\!\!\!/ + q\!\!\!/ + m)}{(p+q)^2 - m^2 + i\epsilon} + \frac{i(p\!\!\!/ - q\!\!\!/' + m)}{(p-q')^2 - m^2 + i\epsilon}\right)\Gamma u^r(p).$$

$\bar{N} + \phi \to \bar{N} + \phi$:

$$i\mathcal{A} = -(-ig)^2 \bar{v}^r(p)\Gamma \left(\frac{i(-\not{p} - \not{q} + m)}{(p+q)^2 - m^2 + i\epsilon} + \frac{i(-\not{p} + \not{q}' + m)}{(p-q')^2 - m^2 + i\epsilon} \right) \Gamma v^{r'}(p').$$

$N + N \to N + N$:

$$i\mathcal{A} = -ig^2 \left(\frac{\bar{u}^{s'}_{q'}\Gamma u^s_{p'} \bar{u}^{r'}_{p'}\Gamma u^r_p}{(q-q')^2 - \mu^2 + i\epsilon} - \frac{\bar{u}^{s'}_{q'}\Gamma u^r_p \bar{u}^{r'}_{p'}\Gamma u^s_q}{(q-p')^2 - \mu^2 + i\epsilon} \right).$$

$N + \bar{N} \to \phi + \phi$:

$N + \bar{N} \to N + \bar{N}$:

$\phi + \phi \to \phi + \phi$ (loop amplitude):

• Attractive Yukawa potential for both $\psi\psi \to \psi\psi$, and also $\psi\bar{\psi} \to \psi\bar{\psi}$. Recall $\mathcal{A}_{NR} = -i\int d^3\vec{r} e^{-i(\vec{p}'-\vec{p})\cdot\vec{r}} U(\vec{r})$. For $\psi\psi \to \psi\psi$, $\mathcal{A}_{NR} \supset -i(-ig)^2(2m)\frac{1}{(\vec{p}-\vec{p}')^2+\mu^2}$ when the spins are unchanged. Gives $U(\vec{r}) = -g^2 e^{-\mu r}/4\pi r$. For $\psi\bar{\psi} \to \psi\bar{\psi}$, amplitude differs by sign, but so does $\bar{v}v$, so again get attractive potential.

• Example $\Gamma = i\gamma_5$, $N + \phi \to N + \phi$, simplify $i\mathcal{A}$. Compute $|\mathcal{A}|^2$ and average over initial spins and sum over final spins. Simplify.

$$i\mathcal{A} = ig^2 \bar{u}^{r'}_{p'} \gamma_5 \left(\frac{\not{p} + \not{q} + m}{(p+q)^2 - m^2 + i\epsilon} + \frac{\not{p} - \not{q}' + m}{(p-q')^2 - m^2 + i\epsilon} \right) \gamma_5 u^r_p,$$

$$i\mathcal{A} = ig^2 \bar{u}^{(r')}(p')\not{q} u^{(r)}(p) F, \qquad F \equiv \left[\frac{1}{2p\cdot q + \mu^2 + i\epsilon} + \frac{1}{2p'\cdot q + \mu^2 + i\epsilon} \right].$$

$$|\mathcal{A}|^2 = g^4 F^2 q_\mu q_\nu Tr[\bar{u}(p')^{r'} \gamma^\mu u(p)^r \bar{u}(p)^r \gamma^\nu u(p)^{r'}].$$

Average over initial spins and sum over final ones (often physically relevant, and it simplifies the expression, using the completeness relations)

$$\begin{aligned} \tfrac{1}{2}\sum_{r,r'} |\mathcal{A}|^2 &= \tfrac{1}{2} g^2 F^2 q_\mu q_\nu Tr[(\not{p}' + m)\gamma^\mu (\not{p} + m)\gamma^\nu] \\ &= 2g^4 F^2 [2(p'\cdot q)(p\cdot q) - p\cdot p' \mu^2 + m^2\mu^2]. \end{aligned}$$