10/8 Lecture outline

$\star$ **Reading for today's lecture: Coleman to end of lecture 4 (p. 37).**

• Last time: symmetries of $\mathcal{L}$ and Noether's theorem. If a variation $\delta\phi_a$ changes $\delta L = \partial_\mu F^\mu$, then it's a symmetry of the action and there is a conserved current: $j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a - F^\mu$.

Example: $x^\mu \to x^\mu + \epsilon^\mu$, $\delta\phi_a = \epsilon^\nu \partial_\nu \phi_a$, $\delta\mathcal{L} = \epsilon^\nu \partial_\nu \mathcal{L}$ (assuming no explicit x dependence). Get $T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\partial_\nu\phi_a - g_{\mu\nu}\mathcal{L}$. Stress energy tensor. Conserved charge is $P_\mu = \int d^3\vec{x} T_{\mu 0}$.

Another example: $\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$, leads to conserved $M_{\mu\rho\sigma} = x_\mu T_{\rho\sigma} - x_\sigma T_{\rho\mu}$. Conserved charge is $M_{\rho\sigma} = \int d^3x M_{0\rho\sigma}$. Conserved angular momentum.

Another example: $\mathcal{L} = \partial_\mu\psi^\dagger\partial^\mu\psi - \mu^2\psi^\dagger\psi$, has symmetry under $\psi \to e^{i\alpha}\psi$. $Q =$ (HW).

• Example from last time: $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$, gives $\Pi = \dot{\phi}$ and $\dot{\Pi} = \nabla^2\phi - m^2\phi$, the Klein-Gordon equation: $(\partial^2 + m^2)\phi = 0$.

• Consider the KG equation in $0+1$ dimensions, i.e. the SHO: $L = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\omega^2\phi^2$, $\Pi = \partial L/\partial\dot{\phi} = \dot{\phi}$. Now quantize: $[\phi, \Pi] = i\hbar$, $[a, a^\dagger] = 1$, $H = \omega(a^\dagger a + \frac{1}{2})$. So a annihilates excitations of energy $\omega \equiv m$, and $a^\dagger$ creates them. In the Heisenberg picture, $\widehat{\phi} = \sqrt{\frac{1}{2\omega}}(ae^{-i\omega t} + a^\dagger e^{i\omega t})$; $\Pi = \dot{\phi} = -i\sqrt{\frac{\omega}{2}}(ae^{i\omega t} - a^\dagger e^{-i\omega t})$. Define $|0\rangle$ s.t. $a|0\rangle = 0$, and $|n\rangle = c_n(a^\dagger)^n|0\rangle$.

• Canonical quantization: generalize QM by replacing $q_a(t) \to \phi(t, \vec{x})$. QM is like QFT in zero spatial dimensions, with the field playing role of position before:

$$[\phi_a(\vec{x}, t), \Pi_b(\vec{y}, t)] = i\delta_{ab}\delta^3(\vec{x} - \vec{y}) \quad \textit{(Equal time commutators).}$$

$$[\phi_a(\vec{x}, t), \phi_b(\vec{y}, t)] = 0.$$

• Quantize the KG field theory in $3+1$ dimensions. Write

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}[a_{\vec{k}}e^{-ikx} + a^\dagger_{\vec{k}}e^{ikx}],$$

$$\Pi(x) = \dot{\phi}(x) = \int \frac{d^3k}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\vec{k}}}{2}}[a_{\vec{k}}e^{-ikx} - a^\dagger_{\vec{k}}e^{ikx}],$$

Then canonical quantization implies that

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = (2\pi)^3\delta^3(\vec{k} - \vec{k}'),$$

creation and annihilation operators, with others vanishing. It will be useful to define $a(k) \equiv \sqrt{2\omega_k} a_{\vec{k}}$, so then the above becomes

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega(k)} [a(k)e^{-ikx} + a^\dagger(k)e^{ikx}],$$

$$[a(k), a^\dagger(k')] = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}'),$$

with the relativistic-invariant measures appearing.

The quantum field is a superposition of creation and annihilation operators. Note also that

$$H = \tfrac{1}{2} \int \frac{d^3k}{(2\pi)^2(2\omega)} \omega(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})),$$

$$\vec{P} = \tfrac{1}{2} \int \frac{d^3k}{(2\pi)^2(2\omega)} \vec{k}(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})),$$

Need to normal order the first term. Define $: AB :$ for operators A and B to mean that the terms are arranged so that the annihilation operators are on the right, so annihilates the vacuum.