10/15 Lecture outline

⋆ **Reading for today's lecture: Luke p. 65-80; Tong p. 35-41.**

• Last time:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega(k)}[a(k)e^{-ikx} + a^\dagger(k)e^{ikx}] \equiv \phi^+(x) + \phi^-(x)$$

where $\phi^\pm$ are positive / negative frequency. Historically, first attempt was to keep just $\phi+$ and regard it as a quantum wavefunction, ψ, with probability $\sim |\psi|^2$. As we saw last time, $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$, but that wouldn't have been true for just $\phi^+(x)$, so there would be information propagating outside the light cone. Moreover, neither $|\phi|^2$ nor $|\phi^+|^2$ can be interpreted as a conserved probability – the relativistic expression $E = \sqrt{\vec{p}^2 + m^2}$ necessarily leads to particle productions. So instead we interpret ϕ as similar to $\vec{x}$ in QM, as a hermitian operator, not a wavefunction.

We showed that $[\phi(x), \phi(y)] = D_1(x-y) - D_1(y-x)$, where

$$\langle 0|\phi(x)\phi(y)|0\rangle = D_1(x-y) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega(k)} e^{-ik(x-y)}.$$

Note that the commutator is a c-number, not an operator. Today, we'll introduce the Feynman propagator, and see that the negative energy solution $E_- = -\sqrt{\vec{p}^2 + m^2}$ can roughly be thought of as being for anti-matter, traveling backwards in time!

• Consider $\mathcal{L} = \frac{1}{2}\partial\phi^2 - \frac{1}{2}m^2\phi^2 - \rho\phi$, where ρ is a classical source. Solve by $\phi = \phi_0 + i\int d^4y D(x-y)\phi(y)$, where ϕ_0 is a solution of the homogeneous KG equation and the green's function $D(x-y)$ satisfies

$$(\partial_x^2 + m^2)D(x-y) = -i\delta^4(x-y).$$

By a F.T., get

$$D_?(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} e^{-ik(x-y)}.$$

Consider the k_0 integral in the complex plane. There are poles at $k_0 = \pm\omega_k$, where $\omega_k \equiv +\sqrt{\vec{k}^2 + m^2}$. There are choices about whether the contour goes above or below the poles, and that's what the ? label indicates.

• Going above both poles gives the retarded green's function, $D_R(x-y)$ which vanishes for $x_0 < y_0$. Considering $x_0 > y_0$, get that

$$D_R(x-y) = \theta(x_0 - y_0)\int \frac{d^3k}{(2\pi)^3 2\omega_k}(e^{-ik(x-y)} - e^{ik(x-y)})$$
$$\equiv \theta(x_0 - y_0)(D(x-y) - D(y-x)) = \theta(x_0 - y_0)\langle[\phi(x), \phi(y)]\rangle,$$

where

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} e^{-ik(x-y)}.$$

This is reasonable: then the $\rho(y)$ source only affects $\phi(x)$ in the future.

Going below both poles gives the advanced propagator, which vanishes for $y_0 < x_0$.

• Feynman propagator: go above the $k_0 = E_k$ pole and below the $k_0 = -E_k$ pole. $-E_k$ pole is heuristically the anti-matter, traveling backward in time. Show that this gives

$$D_F = \theta(x_0 - y_0) D_1(x-y) + \theta(y_0 - x_0) D_1(y-x).$$

Now show

$$D_F(x-y) \equiv \langle T\phi(x)\phi(y)\rangle = \begin{cases} \langle \phi(x)\phi(y)\rangle & \text{if } x_0 > y_0 \\ \langle \phi(y)\phi(x)\rangle & \text{if } y_0 > x_0 \end{cases}.$$

Here T means to time order: order operators so that earliest is on the right, to latest on left. Object like $\langle T\phi(x_1)\ldots\phi(x_n)\rangle$ will play a central role in this class. Time ordering convention can be understood by considering time evolution in $\langle t_f | t_i \rangle$. Evaluate $D_F(x-y)$ by going to momentum space:

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik(x-y)},$$

where $\epsilon \to 0^+$ enforces that we go below the $-\omega_k$ pole and above the $+\omega_k$ pole, i.e. we get $D(x-y)$ if $x_0 > y_0$, and $D(y-x)$ if $x_0 < y_0$, as desired from the definition of time ordering. We'll see that this ensures causality.

• Here's how to remember it: the pole placement is such that the contour can be rotated to be along the imaginary k_0 axis, running from $-i\infty$ to $+i\infty$. This will later tie in with a useful way to treat QFT, by going to Euclidean space via imaginary time. It is something of a technical trick, but there is also something deep about it. Analyticity properties of amplitudes is deeply connected with causality. More later.

• Define contraction of two fields $A(x)$ and $B(y)$ by $T(A(x)B(y)) - :A(x)B(y):$. This is a number, not an operator. Let e.g. $\phi(x) = \phi^+(x) + \phi^-(x)$, where ϕ^+ is the term with annihilation operators and ϕ^- is the one with creation operators (using Heisenberg and Pauli's reversed-looking notation). Then for $x^0 > y^0$ the contraction is $[A^+, B^-]$, and for $y^0 > x^0$ it is $[B^+, A^-]$. So can put between vacuum states to get that the contraction is $\langle TA(x)B(y)\rangle$. For example, in the KG theory the contraction of $\phi(x)$ and $\phi(y)$ is $D_F(x-y)$.