10/22 Lecture outline

$\star$ **Reading for today's lecture: Luke p. 65-80; Tong p. 35-41.**

• Recall the contraction $C(\phi(x)\phi(y))$ (denote it properly in class) is $T(\phi(x)\phi(y)) - N(\phi(x)\phi(y)) = \langle T\phi(x)\phi(y)\rangle = D_F(x-y) = \int \frac{d^4p}{2\pi^4}(i/k^2 - m^2 + i\epsilon)e^{-ik(x-y)}$.

• Last time: Dyson's formula. Compute scattering S-matrices. Consider asymptotic in and out states, with the interaction turned off. Time evolve, with the interaction smoothly turned on and off in the middle.

$$|\psi(t)\rangle = Te^{-i\int d^4x\mathcal{H}_I}|i\rangle.$$

Derive it by solving $i\frac{d}{dt}|\psi(t)\rangle = H_I(t)|\psi(t)\rangle$ iteratively:

$$|\psi(t)\rangle = |i\rangle + (-i)\int_{-\infty}^{t} dt_1 H_I(t_1)|\psi(t_1)\rangle$$

$$|\psi(t_1)\rangle = |i\rangle + (-i)\int_{-\infty}^{t_1} dt_2 H_I(t_2)|\psi(t_2)\rangle$$

etc where $t_1 > t_2$, and then symmetrize in t_1 and t_2 etc., which is what the T time ordering does.

• Now use Wick's theorem:

$$\begin{aligned} T(\phi_1\ldots\phi_n) &=: \phi_1\ldots\phi_n : + \sum_{contractions} : \phi_1\ldots\phi_n : \\ &=: e^{\frac{1}{2}\sum_{i,j=1}^{n} C(\phi_i\phi_j)\frac{\partial}{\partial\phi_i}\frac{\partial}{\partial\phi_j}}\phi_1\ldots\phi_n \end{aligned} \quad :$$

(where C is the contraction symbol) to get rid of the time ordered products.

Prove Wick's theorem by iteration: define the RHS as $W(\phi_1\ldots\phi_n)$ and we assume $T(\phi_2\ldots\phi_n) = W(\phi_2\ldots\phi_n)$ and want to prove then that $T(\phi_1\ldots\phi_n) = W(\phi_1\ldots\phi_n)$. WLOG, take $t_1 > t_2\ldots t_n$ so $T(\phi_1\ldots\phi_n) = \phi_1 T(\phi_2\ldots\phi_n) = \phi_1 W(\phi_2\ldots\phi_n) = \phi_1^- W + W\phi_1^+ + [\phi_1^+, W]$. The first two terms are normal ordered and give all contractions not involving ϕ_1, while the last gives all normal ordered contractions involving ϕ_1.

So note that

$$\langle T(\phi_1\ldots\phi_n)\rangle \begin{cases} 0 & \text{for } n \text{ odd} \\ \sum_{\text{fullycontracted}} & \text{for } n \text{ even.} \end{cases}$$

• Thereby compute probability amplitude for a given process

$$\langle f|(S-1)|i\rangle = \langle f|Te^{-i\int d^4x\mathcal{H}_I(x)}|i\rangle \equiv i\mathcal{A}_{fi}(2\pi)^4\delta^{(4)}(p_f - p_i).$$

The initial states have momenta $p_1 \dots p_n$ and the final states have momenta $q_1 \dots q_m$. Need to strip off the momentum conserving delta function to get the amplitude.

• Look at some examples, and connect with Feynman diagrams. As a first, simple example consider the above theory, with $H_{int} = \int d^3x g\phi\psi^\dagger\psi$. Use $\phi \sim a + a^\dagger$ for "mesons," $\psi \sim b + c^\dagger$, and $\psi^\dagger \sim b^\dagger + c$. We'll say that b annihilates a nucleon N and $c^\dagger$ creates an anti-nucleon $\bar{N}$. Conservation law, conserved charge $Q = N_b - N_c$.

Examples of states:

$$|\phi(p)\rangle = a^\dagger(p)|0\rangle, \qquad |N(p)\rangle = b^\dagger(p)|0\rangle, \qquad |\bar{N}(p)\rangle = c^\dagger(p)|0\rangle.$$

Note then e.g.

$$\langle 0|\phi(x)|\phi(p)\rangle = e^{-ip\cdot x}, \qquad \langle 0|\psi(x)|N(p)\rangle = e^{-ip\cdot x}, \qquad \langle 0|\psi^\dagger(x)|N(p)\rangle = 0.$$

Example: meson decay. $|i\rangle = a^\dagger(p)|0\rangle$, $|f\rangle = b^\dagger(q_1)c^\dagger(q_2)|0\rangle$. Compute $\langle f|S|i\rangle = -ig(2\pi)^4\delta^4(p - q_1 - q_2)$ to $\mathcal{O}(g)$, i.e. $\mathcal{A} = -g$. Probability $\sim g^2$.

Comment: draw pictures to illustrate a $\sim g^3$ correction, with 1 loop. In general, amplitudes scale like $(g^2/16\pi^2)^L$ where L is the number of loops. But we'll see that loops lead to divergent momenta integrals, eg. $\int^\Lambda d^4k/k^2 - m^2 \sim \Lambda^2$. How to handle this will be deferred to next quarter...