10/1 Lecture outline

$\star$ **Reading for today's lecture: Coleman lecture 0 and 1, Luke chapter 1, Tong chapter 0 and 1.**

• Relativistic QM? In QM we have $\psi(\vec{x},t) = \langle x|e^{-iHt/\hbar}|\psi\rangle$, with $\vec{x}$ the eigenvalue of an operator, vs t a parameter – different treatment of space and time problematic with relativity. Also $\int d^3\vec{x}|\psi|^2 = 1$ expresses particle conservation, which is not true in nature, e.g. $p+p \to$ lots of stuff at the LHC. Particle number can change; can't have a relativistic single particle QM. Let $\lambda_c \equiv \hbar/mc$ be the Compton wavelength, as opposed to $\lambda_{DB} \equiv h/p$. E.g. for an electron get $\lambda_c \sim 4\times 10^{-11} cm \ll 1\mathring{A}$. Ordinary QM $\approx$ OK when $\lambda_{DB} \gg \lambda_c$, i.e. non-relativisitic velocities. Particle creation when we try to localize on distances $L \sim \lambda_c$.

Issues: Can't define $\vec{X}_{op}$ or $|\vec{x}\rangle$. In QM treat $\vec{X}$ as an operator and t as a parameter, in a relativistic theory they must be treated symmetrically. In QM, observables, e.g. L_z, aren't attached to their location, so can have problems with causality. Want something like $[O_1(x_1), O_2(x_2)] = 0$ for spacelike separations, $(x_1 - x_2)^2 < 0$.

• Solution: quantum field theory. Each elementary particle, e.g. electron, is replaced with fluctuations of a local field $\phi(x^\mu)$. We treat $x^\mu = (t, \vec{x})$ as parameters, quantize ϕ rather than x^μ. Agrees with the fact that all electrons are the same. Whether here or on the other side of the universe, electrons are the same kind of blip of the electron field, which fills the universe.

• Conventions: $\hbar = c = 1$, mostly minus metric $g_{\mu\nu}$, e.g. $\partial_\mu = (\partial_t, \vec{\nabla})$, $\partial_\mu\partial^\mu \equiv \partial^2 = \partial_t^2 - \nabla^2$.

• Recall from QM: $[q_r, p_s] = i\hbar\delta_{rs}$. Momentum eigenstates $|\vec{p}\rangle$, s.t. $\langle\vec{p}|\vec{p}'\rangle = \delta_{\vec{p},\vec{p}'}$, form a complete basis for 1-particle Hilbert space, $\mathbf{1} = \int \frac{d^3\vec{p}}{(2\pi)^3}|\vec{p}\rangle\langle\vec{p}|$. Position eigenstates $\vec{X}|\vec{x}\rangle = \vec{x}|\vec{x}\rangle$, and $\langle\vec{x}|\vec{p}\rangle = Ce^{i\vec{p}\cdot\vec{x}/\hbar}$, where C is a conventional normalization coefficient, e.g. take $C = (2\pi)^{-3/2}$. Note that $e^{i\vec{P}\cdot\vec{a}/\hbar}$ generates spatial translations: $|\vec{x}+\vec{a}\rangle = e^{i\vec{P}\cdot\vec{a}}|\vec{x}\rangle$. Note that $e^{-iHa_0/\hbar}$ generates time translations, $e^{-iHa_0/\hbar}|\psi(t)\rangle = |\psi(t+a_0)\rangle$. These statements already show some Lorentz invariance: $e^{-iPa/\hbar}$ generates space-time translations. In position space, we replace $P_\mu \to i\hbar\partial_\mu$, so the plane-wave eigenstates are $\langle\vec{x}|e^{-iHt/\hbar}|\vec{p}\rangle = Ce^{ipx/\hbar}$.

• We'll see that we'll keep some of this structure, in particular momentum eigenstates, but have to trash the notion of position eigenstates. As a technical point, we'll need to change our normalization of the momentum eigenstates to work with Lorentz-invariant quantities. Note that $d^4k\delta(k^2-m^2)\theta(k_0) \to \frac{d^3k}{2\omega(k)}$ upon doing the k_0 integral. So normalize $\langle k'|k\rangle = (2\pi)^3 2\omega(k)\delta^3(\vec{k}-\vec{k}')$, with $|k\rangle \equiv \sqrt{(2\pi)^3 2\omega_k}|\vec{k}\rangle$.

• Q: "What goes wrong if we just do the single-particle S.E. with $H_{rel} = \sqrt{\vec{p}^2+m^2}$?" A: many things. One question is how to make sense of $H = \sqrt{\vec{P}^2+m^2}$, where $\vec{P}$ is an operator. How do you take a square-root of an operator? This question led Dirac to his equation, which describes fermions. Requires anti-matter to make sense of it. Can't have well-defined single-particle states. Here let's illustrate another issue: correlators outside the forward light-cone. Recall that any point outside the light-cone can be mapped by a Lorentz transformation to any other point, and that signals sent outside the light-cone can then be used to transmit information back in time, violating causality. As far as we're aware, that is illegal. Here's how to show it:

Start with $|\psi(t=0)\rangle = |\vec{x}=0\rangle$. Compute

$$\begin{aligned}
\langle \vec{x}|\psi(t)\rangle = \langle \vec{x}|e^{-iHt}|\vec{x}=0\rangle &= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} e^{-i\sqrt{\vec{p}^2+m^2}t} \\
&= -\frac{i}{(2\pi)^2 r}\int_{-\infty}^{\infty} pdpe^{ipr}e^{-i\sqrt{p^2+m^2}t} \\
&= \frac{ie^{-mr}}{2\pi^2 r}\int_m^{\infty} dzze^{-(z-m)r}\sinh(\sqrt{z^2-m^2}t)
\end{aligned}$$

The last step is by deforming the contour in the complex p plane, and getting contributions along the branch cut in the UHP, with $z = -ip$; the contribution along the big semi-circle at infinity vanishes for $r > t$. The integral is positive, so non-vanishing outside the forward light cone: acausal, with causality recovered as an approximation for $r \gg m$. In QFT, the difference will be antiparticles to the rescue! The antiparticle contribution is added, and cancels the acausality. Must give up on purely single-particle states in the relativistic quantum realm.