11/4 Lecture outline

$\star$ **Reading for this week's lecture: Coleman lecture notes pages 109-139.**

• Continue from last time. Compute probabilities by squaring the S-maxtrix amplitudes, but have to be careful with the delta functions, since squaring the delta functions would give nonsense. In QM we found $\dot{P}_{i\to f} = 2\pi|\langle f|H_{int}|i\rangle|^2\rho(E)$ "Fermi's golden rule."

Phase space factors. Put the system in a box of volume V. The momenta are quantized and, as usual, there are $Vd^3\vec{k}/(2\pi)^3$ states with $\vec{k}$ in the range $d^3\vec{k}$. Interested in computing probabilities, $P = |\langle f|i\rangle|^2/\langle f|f\rangle\langle i|i\rangle$. Use $\langle k|k\rangle = (2\pi)^3 2\omega\delta^3(0)$ (since $\int d^3\vec{x}e^{i\vec{p}\cdot\vec{x}} = (2\pi)^3\delta^3(\vec{p})$ gives for $\vec{p}=0$: $\int d^3x 1 = V$) and replace $(2\pi)^3\delta^3(0) \to V$. Put these normalization factors into correct normalization of initial and final states $|\widehat{i}\rangle = |i\rangle/\sqrt{\langle i|i\rangle}$):

$$\langle\widehat{f}|(S-1)|\widehat{i}\rangle_{VT} = i\mathcal{A}_{fi}^{VT}(2\pi)^4\delta^4(p_F - p_I)\prod_f \frac{1}{\sqrt{2\omega_k V}}\prod_i \frac{1}{\sqrt{2\omega_k V}},$$

where the factors account for the relativistic normalization of the states. Squaring, with the replacement $(2\pi^4\delta^4(p))^2 \to VT(2\pi)^4\delta^4(p)$ (since $\int d^4xe^{i0\cdot x} = VT$) get that the probability per unit time is

$$|\mathcal{A}_{fi}|^2 VD\prod_i \frac{1}{2E_iV},$$

where D also accounts for the $Vd^3p_f/(2\pi)^3$ available final states in that $\vec{p}_f$ interval

$$D = (2\pi)^4\delta^4(p_F - p_I)\prod_f \frac{d^3p_f}{(2\pi)^3 2E_f}.$$

Verify units: $[\dot{P}] = 2(4 - n_{i,tot} - n_{f,tot}) - 3 + 2n_{i,tot} - 4 + 2n_{f,tot} = 1$, good.

Decays: differential decay probability per unit time: $d\Gamma = \frac{1}{2M}|\mathcal{A}_{fi}|^2 D$. Integrate over all possible final states to get $\Gamma = 1/\tau$ where τ is the lifetime.

Cross sections: the number of scatterings per unit time is $dN = Fd\sigma$, where F is the flux. So

$$d\sigma = \frac{\mathcal{A}_{fi}^2}{4E_1E_2V}\ D\ \frac{V}{|\vec{v}_1 - \vec{v}_2|},$$

where the last factor is from dividing by the flux $F = |\vec{v}_{rel}|\rho$, using that the particle density is $\rho = 1/V$ (get V/V^2 for colliding two beams).

Note that this is relativistic. Write $dNdt = (d\sigma|\vec{v}_1 - \vec{v}_2|\rho_1\rho_2)(Vdt)$, the LHS is the number of collisions, which should be the same in any frame, and the factor (Vdt) on the

RHS is relativistically invariant. For simplicity we take $\vec{v}_1$ and $\vec{v}_2$ to be parallel, $\vec{v}_1 \times \vec{v}_2 = 0$. We want $d\sigma$ to be defined to be the cross section in the rest frame of one of the particles, so we want to define it to be boost invariant. So we need to show that $|\vec{v}_1 - \vec{v}_2|\rho_1\rho_2$ is boost invariant; in the rest frame of particle 2 it reduces to $v_{rel}\rho_1\rho_2$, which is what we want. Let's just check it. Under a boost to a frame with relative velocity u (taken along the direction of $\vec{v}_1$ and $\vec{v}_2$, we have $v_i \to (v_i + u)/(1 + v_i u)$ and $\rho_i \to \rho_i \gamma_u (1 + v_i u)$ (recall $J_i^\mu = \rho_i(1, \vec{v}_i)$ transforms as a 4-vector), so $|\vec{v}_1 - \vec{v}_2|\rho_1\rho_2$ is indeed invariant. For our application, we define $\rho_i = 1/V$ in the lab frame.

Two body final states (in CM frame): $D = \int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^3(\vec{p}_1 + \vec{p}_2)\delta(E_1 + E_2 - E_T)$ gives

$$D = \int \frac{1}{(2\pi)^3 4E_1E_2} p_1^2 dp_1 d\Omega_1 (2\pi)\delta(E_1 + E_2 - E_T).$$

Using $E_1 = \sqrt{p_1^2 + m_1^2}$ and $E_2 = \sqrt{p_1^2 + m_2^2}$ get $\partial(E_1 + E_2)/\partial p_1 = p_1 E_T / E_1 E_2$ and finally $D = p_1 d\Omega_1 / 16\pi^2 E_T$. This should be divided by 2! (more generally, n!) if the final states are identical.

- Example. For $\mu^2 > 4m^2$, consider $\phi \to \bar{N}N$ decay. $\mathcal{A} = -g$, and get

$$\Gamma = \frac{g^2}{2\mu} \frac{p_1}{16\pi^2 \mu} \int d\Omega_1 = \frac{g^2}{8\pi\mu^2} \frac{\sqrt{\mu^2 - 4m^2}}{2},$$

where the last factor is p_1.

For $2 \to 2$ scattering in the CM frame,

$$d\sigma = \frac{|\mathcal{A}|^2}{4E_1E_2} \frac{p_f d\Omega_1}{16\pi^2 E_T} \frac{1}{|\vec{v}_1 - \vec{v}_2|} = \frac{|\mathcal{A}|^2 p_f d\Omega_1}{64\pi^2 p_i E_T^2}$$

where we used $|\vec{v}_1 - \vec{v}_2| = p_1(E_1^{-1} + E_2^{-2}) = p_i E_T / E_1 E_2$ in the CM frame, and p_i is the magnitude of the initial 3-momentum, and p_f is that of the final momentum. Note that they can be different if the initial and final states are of particles of different masses, e.g. $e^+e^- \to \mu^+\mu^-$.

- Summary:

$$d\Gamma_{1\to 2} = \frac{|\mathcal{A}|^2 D_{2-body}}{2M}$$

$$d\sigma_{2\to 2} = \frac{|\mathcal{A}|^2}{4E_1E_2} D_{2-body} \frac{1}{|\vec{v}_1 - \vec{v}_2|}$$

$$D_{2-body(CM)} = \frac{p_1 d\Omega_1}{16\pi^2 E_{CM}} \qquad \text{(divide by 2! if identical final states).}$$

Examples: For $\mu^2 > 4m^2$, consider $\phi \to \bar{N}N$ decay. $\mathcal{A} = -g$, and get

$$\Gamma = \frac{g^2}{2\mu}\frac{p_1}{16\pi^2\mu}\int d\Omega_1 = \frac{g^2}{8\pi\mu^2}\frac{\sqrt{\mu^2 - 4m^2}}{2},$$

where the last factor is p_1. For $2 \to 2$ scattering in the CM frame,

$$d\sigma = \frac{|\mathcal{A}|^2}{4E_1E_2}\frac{p_f d\Omega_1}{16\pi^2 E_T}\frac{1}{|\vec{v}_1 - \vec{v}_2|} = \frac{|\mathcal{A}|^2 p_f d\Omega_1}{64\pi^2 p_i E_T^2}$$

where we used $|\vec{v}_1 - \vec{v}_2| = p_1(E_1^{-1} + E_2^{-2}) = p_i E_T / E_1 E_2$ in the CM frame, and p_i is the magnitude of the initial momentum, and p_f is that of the final momentum.