11/9 Lecture outline

⋆ **Reading for today's lecture: Coleman lecture notes pages 140-175.**

• Brief introduction to a better description of QFT and perturbation theory.]Define the true vacuum $|\Omega\rangle$ such that $H|\Omega\rangle = 0$, and $\langle\Omega|\Omega\rangle = 1$. The true vacuum of an interacting QFT is a complicated beast – it can be thought of roughly as a soup of particle-antiparticle states – it can not be solved for solved for exactly. (Progress: in classical mechanics, can solve 2 body problem exactly, but ≥ 3 body only approximately; in GR, can solve 1 body problem exactly, but ≥ 2 body only approximately; in QM can generally solve even only 1-body problem only approximately, but at least the 0-body problem is trivial; in QFT, even the 0-body problem is not exactly solvable.)

Define Green functions or correlation functions by

$$G^{(n)}(x_1, \dots x_n) = \langle\Omega|T\phi_H(x_1)\dots\phi_H(x_n)|\Omega\rangle,$$

where $\phi_H(x)$ are the full Heisenberg picture fields, using the full Hamiltonian.

Now show that

$$G^{(n)}(x_1 \dots x_n) = \frac{\langle 0|T\phi_{1I}(x_1)\dots\phi_{nI}(x_n)S|0\rangle}{\langle 0|S|0\rangle},$$

where $|0\rangle$ is the vacuum of the free theory, and ϕ_{iI} are interaction picture fields, and the S in the numerator and denominator gives the interaction-Hamiltonian time evolution from $-\infty$ to x_n, then from x_n to x_{n-1} etc and finally to $t = +\infty$. To show it, take $t_1 > t_2 \dots > t_n$ and put in factors of $U_I(t_a, t_b) = T\exp(-i\int_{t_a}^{t_b} H_I)$ to convert ϕ_I to ϕ_H, using $\phi_H(x_i) = U_I(t_i, 0)^\dagger\phi_I(x_i)U_I(t_i, 0)$. Get $\langle 0|U_I(\infty, t_1)\phi_H(t_1)\dots\phi_H(t_n)U_I(t_n, -\infty)|0\rangle$, and U_I at ends can be replaced with full $U(t_1, t_2)$, since $H_0|0\rangle = 0$ anyway. Now use

$$\begin{aligned}\langle\Psi|U(t,-\infty)|0\rangle &= \langle\Psi|U(t,-\infty)\left(|\Omega\rangle\langle\Omega| + \sum\int |n\rangle\langle n|\right)|0\rangle \\ &= \langle\Psi|\Omega\rangle\langle\Omega|0\rangle + \lim_{t'\to-\infty}\sum\int e^{iE_n(t'-t)}\langle\Psi|n\rangle\langle n|0\rangle \\ &= \langle\Psi|\Omega\rangle\langle\Omega|0\rangle\end{aligned}$$

where 1 was inserted as a complete set of states, including the vacuum and single and multiparticle states, including integrating over their momenta, but the wildly oscillating factor kills all those terms. (Riemann-Lebesgue lemma: $\lim_{t\to\infty}\int d\omega f(\omega)e^{i\omega t} = 0$ for nice $f(\omega)$) The result follows upon doing the same for the denominator.

The $\langle 0|S|0\rangle$ in the denominator eliminates the vacuum bubble diagrams. So we have

$$G^{(n)}(x_1, \ldots x_n) = \sum \text{Feynman graphs without vacuum bubbles.}$$

• Example: $G^{(4)}(x_1, x_2, x_3, x_4)$ in $\lambda\phi^4/4!$ theory. For each line from x to y, get a factor of $\Delta_F(x-y)$, and for each vertex at y get $-i\lambda \int d^4y$. Includes connected and disconnected diagrams. Disconnected ones will go away when computing S-matrix elements.

• It's more convenient often to work in momentum space,

$$\widetilde{G}^{(n)}(p_1, \ldots p_n) = \int \prod_{i=1}^{n} d^4x_i e^{-ip_i x_i} G^{(n)}(x_1 \ldots x_n).$$

Similar to what we computed before to get S-matrix elements, but the external legs include their propagators, and the external momenta are not on-shell.

• From Green functions $\widetilde{G}^{(n)}(p_1, \ldots, p_n)$, computed with external leg propagators, allowed to be off-shell, to S-matrix elements. E.g.

$$\langle k_3, k_4|S-1|k_1k_2\rangle = \prod_{n=1}^{4} \frac{k_n^2 - m_n^2}{i\sqrt{Z}} \widetilde{G}(-k_3, -k_4, k_1, k_2),$$

where the factors are to amputate the external legs. Consider for example $\widetilde{G}^{(4)}(k_1, k_2, k_3, k_4)$ for 4 external mesons in our meson-nucleon toy model. The lowest order contribution is at $\mathcal{O}(g^0)$ and is

$$(2\pi)^4\delta^{(4)}(k_1+k_4)\frac{i}{k_1^2-\mu^2+i\epsilon}(2\pi)^4\delta^{(4)}(k_2+k_3)\frac{i}{k_2^2-\mu^2+i\epsilon} + 2 \text{ permutations.}$$

This is the -1 that we subtract in $S-1$, and indeed would not contribute to $2 \to 2$ scattering using the above formula, because it is set to zero by $\prod_{n=1}^{4}(k_n^2 - m_n^2)$ when the external momenta are put on shell. To get a non-zero result, need a $\widetilde{G}^{(4)}$ contribution with 4 external propagators, which we get e.g. at $\mathcal{O}(g^4)$ with an internal nucleon loop.

• Account for bare vs full interacting fields. Let $|k\rangle$ be the physical one-meson state of the full interacting theory, normalized to $\langle k'|k\rangle = (2\pi)^3 2\omega_k \delta^{(3)}(\vec{k}' - \vec{k})$. Then

$$\langle k|\phi(x)|\Omega\rangle = \langle k|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|\Omega\rangle = e^{ik\cdot x}\langle k|\phi(0)|\Omega\rangle \equiv e^{ik\cdot x} Z_\phi^{1/2}.$$

Can rescale the fields, s.t. $\langle k|\phi_R(x)|\Omega\rangle = e^{-ik\cdot x}$. The LSZ formula is:

$$\langle q_1 \ldots q_n|S-1|k_1 \ldots k_m\rangle = \prod_{a=1}^{n} \frac{q_a^2 - m_a^2}{i\sqrt{Z}} \prod_{b=1}^{m} \frac{k_b^2 - m_b^2}{i\sqrt{Z}} \widetilde{G}^{(n+m)}(-q_1, \ldots - q_n, k_1, \ldots k_m),$$

where the Green function is for the Heisenberg fields in the full interacting vacuum.

To derive the LSZ formula, consider wave packets, with some profile $F(\vec{k})$, and $f(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} F(\vec{k}) e^{-ik\cdot x}$, where we define $k_0 = \sqrt{\vec{k}^2 + \mu^2}$, so $f(x)$ solves the KG equation. Now define

$$\phi^f(t) = i \int d^3\vec{x} (\phi(\vec{x}, t)\partial_0 f(\vec{x}, t) - f(\vec{x}, t)\partial_0 \phi(\vec{x}, t)).$$

Note that, since $f(x)$ satisfies the KG equation, can show

$$i \int d^4x f(x)(\partial^2 + \mu^2)\phi(x) = -\int dt \frac{\partial}{\partial t}\phi^f(t) = -\phi^f(t)|_{-\infty}^{\infty}.$$

(Sign choice nice for making in states.) Show that $\phi^f(t)$ makes single particle wave packets from the vacuum, $\langle k|\phi^f(t)|\Omega\rangle = F(\vec{k})$. Can similarly show (because of a relative minus sign), $\langle \Omega|\phi^f(t)|k\rangle = 0$, and $\langle n|\phi^f(t)|\Omega\rangle = \frac{\omega_{p_n} + p_n^0}{2\omega_{p_n}} F(\vec{p}_n) e^{-i(\omega_{p_n} - p_n^0)t} \langle n|\phi(0)\Omega\rangle$, where $\omega_{p_n} \equiv \sqrt{\vec{p}_n^2 + \mu^2}$, which has $\omega_{p_n} < p_n^0$ for any multiparticle state. So $\lim_{t\to\pm\infty} \langle \psi|\phi^f(t)|\Omega\rangle = \langle \psi|f\rangle + 0$, where the multiparticle states contributions sum to zero using the Riemann-Lebesgue lemma.

Make separated in states: $|f_n\rangle = \prod \phi^{f_n}(t_n)|\Omega\rangle$, and out states $\langle f_m| = \langle \Omega| \prod (\phi^{f_m})^\dagger(t_m)$, with $t_n \to -\infty$ and $t_m \to +\infty$. Then show

$$\langle f_m|S - 1|f_n\rangle = \int \prod_n d^4x_n f_n(x_n) \prod_m d^4x_m f_m(x_m)^* \prod_r i(\partial_r^2 + m_r^2) G(x_n, x_m).$$

Take $f_i(x) \to e^{-ik_i x_i}$ at the end. Show that all the $t \to \pm\infty$ do the right thing to give the in and out states, thanks to various cancellations, using $\lim_{t\to\pm\infty} \langle \Psi|\phi^f(t)|\Omega\rangle = \langle \Psi|f\rangle$.

• On to fermions! Consider more generally Lorentz transformations. Under lorentz transformations $x^\mu \to x^{\mu'} = \Lambda^\mu_\nu x^\nu$, scalar fields transform as $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$. Vector fields transform as $A^\mu \to \Lambda^\mu_\nu A^\nu(\Lambda^{-1}x)$. Generally, $\phi^a \to D[\Lambda]^a_b \phi^b(\Lambda^{-1}x)$, where $D[\Lambda]$ is a rep of the Lorentz group, $D[\Lambda_1]D[\Lambda_2] = D[\Lambda_1\Lambda_2]$.