9/30 Lecture outline

⋆ **Reading for today's lecture: Coleman to end of lecture 4 (p. 37).**

• Recall where we left off: show that $\langle x^\mu|0^\mu\rangle = \int \frac{d^3k}{(2\pi)^3}e^{-ip\cdot x}$ is not zero even for spacelike separation, $x^2<0$:

$$\begin{aligned}\langle\vec{x}|\psi(t)\rangle = \langle\vec{x}|e^{-iHt}|\vec{x}=0\rangle &= \int \frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot\vec{x}}e^{-i\sqrt{\vec{p}^2+m^2}t}\\ &= -\frac{i}{(2\pi)^2 r}\int_{-\infty}^{\infty} pdpe^{ipr}e^{-i\sqrt{p^2+m^2}t}\\ &= \frac{ie^{-mr}}{2\pi^2 r}\int_m^\infty dzze^{-(z-m)r}\sinh(\sqrt{z^2-m^2}t)\end{aligned}$$

The last step is by deforming the contour in the complex p plane, and getting contributions along the branch cut in the UHP, with $z=-ip$; the contribution along the big semi-circle at infinity vanishes for $r>t$. The integral is positive, so non-vanishing outside the forward light cone: acausal, with causality recovered as an approximation for $r\gg m$. In QFT, the difference will be antiparticles to the rescue! The antiparticle contribution is added, and cancels the acausality. Must give up on purely single-particle states in the relativistic quantum realm.

Resolution: we can't have position eigenstates and operators. Replace particles with ripples of quantum field, e.g. $\phi(t,\vec{r})$, as we did for the case of the SHO, which again we can reinterpret as a QFT in $d=0+1$ dimensions, with $q(t)$ playing role of $\phi(t)$.

• Multiparticle warmup: recall SHO, $[a,a^\dagger]=1$, and states. Recall $\mathbf{1}=|0\rangle\langle 0|+\sum_{n=1}^\infty |n\rangle\langle n|$, sum over phonon occupation number states.

• QFT in $d=3+1$ dimensions, replace particles with ripples of quantum field, e.g. $\phi(t,\vec{r})$. Mention QFT in $d=0+1$ dimensions is QM, with $q(t)$ playing role of $\phi(t)$.

• Convenient to work in momentum space. Sometimes it's mathematically convenient to think about the theory in a box, to make momenta discrete. Then the integrals become sums, and the delta functions become Kroneker deltas. Can then count how many excitations of each momenta. Fock space description, like counting the excitation level of the SHO. Like, there, we'll introduce creation and annihilation operators.

• Classical and quantum particle mechanics, $L(q_a,\dot{q}_a,t)$, $p_a=\partial L/\partial\dot{q}_a$, $\dot{p}_a=\partial L/\partial q_a$, $H=\sum_a p_a\dot{q}_a - L$. Get quantum theory by replacing Poisson brackets with commutators, $[q_a(t),p_b(t)]=i\delta_{ab}$. Recall $O_H(t)=e^{iHt}O_Se^{-iHt}$ and $i\frac{d}{dt}O_H(t)=[O_H(t),H]$.

• Classical field theory. E.g. scalars $\phi_a(t,\vec{x})$, with $S = \int d^4x\mathcal{L}(\phi_a, \partial_\mu\phi_a)$. Then $\Pi_a^\mu = \partial\mathcal{L}/\partial(\partial_\mu\phi_a)$, and E.L. eqns $\partial\mathcal{L}/\partial\phi_a = \partial_\mu\Pi_a^\mu$. Define $\Pi_a \equiv \Pi_a^0$. $H = \int d^3x(\Pi\dot{\phi}_a - \mathcal{L}) = \int d^3x\mathcal{H}$.

• Example: $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$, gives $\Pi = \dot{\phi}$ and $\dot{\Pi} = \nabla^2\phi - m^2\phi$, the Klein-Gordon equation: $(\partial^2+m^2)\phi = 0$. Can't interpret ϕ as a probability wavefunction because of solutions $E = \pm\sqrt{\vec{p}^2 + m^2}$.

But we'll see that the KG equation is fine as a quantum field theory. As a classical field theory, write general classical solution as

$$\phi_{cl}(x) = \int \frac{d^3k}{(2\pi)^3(2\omega(k))}[a_{cl}(k)e^{-ikx} + a_{cl}^*(k)e^{ikx}],$$

where $a_{cl}(k)$ are classical constants of integration, determined by the initial conditions.

• The normalization of the momentum space integral is chosen to be relativistically nice: it's Lorentz invariant: $d^3k/\omega = d^3k'/\omega'$. Here's why: $d^4k\delta(k^2 - m^2)\theta(k_0) \to \frac{d^3k}{2\omega(k)}$ upon doing the k_0 integral. So normalize $\langle k'|k\rangle = (2\pi)^3 2\omega(k)\delta^3(\vec{k} - \vec{k}')$, with $|k\rangle \equiv \sqrt{(2\pi)^3 2\omega_k}|\vec{k}\rangle$.

• Important aspect of classical or quantum field theory: continuous symmetries of $\mathcal{L}$ lead to conservation laws, via Noether's theorem. If a variation $\delta\phi_a$ changes $\delta\mathcal{L} = \partial_\mu F^\mu$, then it's a symmetry of the action and there is a conserved current: $j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a - F^\mu$.

Example: $x^\mu \to x^\mu + \epsilon^\mu$, $\delta\phi_a = \epsilon^\nu\partial_\nu\phi_a$, $\delta\mathcal{L} = \epsilon^\nu\partial_\nu\mathcal{L}$ (assuming no explicit x dependence). Get $T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\partial_\nu\phi_a - g_{\mu\nu}\mathcal{L}$. Stress energy tensor. Conserved charge is $P_\mu = \int d^3\vec{x}T_{\mu 0}$.

Another example: $\Lambda_\nu^\mu = \delta_\nu^\mu + \omega_\nu^\mu$, leads to conserved $M_{\mu\rho\sigma} = x_\mu T_{\rho\sigma} - x_\sigma T_{\rho\mu}$. Conserved charge is $M_{\rho\sigma} = \int d^3x M_{0\rho\sigma}$. Conserved angular momentum.