10/5 Lecture outline

⋆ **Reading for today's lecture: Coleman to end of lecture 4 (p. 37).**

• Last time: symmetries of $\mathcal{L}$ and Noether's theorem. If a variation $\delta\phi_a$ changes $\delta\mathcal{L} = \partial_\mu F^\mu$, then it's a symmetry of the action and there is a conserved current: $j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a - F^\mu$.

Translation invariance: $x^\mu \to x^\mu + \epsilon^\mu$, $\delta\phi_a = \epsilon^\nu\partial_\nu\phi_a$, $\delta\mathcal{L} = \epsilon^\nu\partial_\nu\mathcal{L}$ (assuming no explicit x dependence). Get $T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\partial_\nu\phi_a - g_{\mu\nu}\mathcal{L}$. Stress energy tensor. Conserved charge is $P_\mu = \int d^3\vec{x}T_{\mu 0}$. Another example: $x^{\mu'} = \Lambda^{\mu'}_\nu x^\nu$ Lorentz boost and rotation symmetry leads to conservation of angular momentum. Write $\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$, leads to conserved $M_{\mu\rho\sigma} = x_\mu T_{\rho\sigma} - x_\sigma T_{\rho\mu}$. Conserved charge is $M_{\rho\sigma} = \int d^3x M_{0\rho\sigma}$.

• Apply to $\mathcal{L}_{KG} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$. Get $T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\lambda\phi\partial^\lambda\phi + \frac{1}{2}m^2\phi^2\eta_{\mu\nu}$. So

$$H = \int d^3x\mathcal{H}, \qquad \vec{P} = \int d^3x\vec{\mathcal{P}}.$$

$$\mathcal{H} = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2, \qquad \vec{\mathcal{P}} = \dot{\phi}\,\nabla\phi.$$

• Recall from last week: SHO = KG equation in $0+1$ dimensions, i.e. the SHO: $L = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\omega^2\phi^2$, $\Pi = \partial L/\partial\dot{\phi} = \dot{\phi}$. Now quantize: $[\phi, \Pi] = i\hbar$, $[a, a^\dagger] = 1$, $H = \omega(a^\dagger a + \frac{1}{2})$. Heisenberg picture, $\widehat{\phi} = \sqrt{\frac{1}{2\omega}}(ae^{-i\omega t} + a^\dagger e^{i\omega t})$; $\Pi = \dot{\phi} = -i\sqrt{\frac{\omega}{2}}(ae^{i\omega t} - a^\dagger e^{-i\omega t})$. Define $|0\rangle$ s.t. $a|0\rangle = 0$, and $|n\rangle = c_n(a^\dagger)^n|0\rangle$.

• Canonical quantization: generalize QM by replacing $q_a(t) \to \phi(t, \vec{x})$. It's conjugate momentum is $\Pi \equiv \partial\mathcal{L}/\partial\dot{\phi}$. The theory is quantized by replacing ϕ and Π with operators (sometimes we'll give them hats, but usually won't bother), satisfying

$$[\phi_a(\vec{x}, t), \Pi_b(\vec{y}, t)] = i\delta_{ab}\delta^3(\vec{x} - \vec{y}) \quad \textit{(Equal time commutators).}$$

$$[\phi_a(\vec{x}, t), \phi_b(\vec{y}, t)] = 0.$$

• Quantize the KG field theory in $3+1$ dimensions. Write

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}[a_{\vec{k}}e^{-ikx} + a^\dagger_{\vec{k}}e^{ikx}],$$

$$\Pi(x) = \dot{\phi}(x) = \int \frac{d^3k}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\vec{k}}}{2}}[a_{\vec{k}}e^{-ikx} - a^\dagger_{\vec{k}}e^{ikx}],$$

Then canonical quantization implies that

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = (2\pi)^3\delta^3(\vec{k}-\vec{k}'),$$

i.e. they're creation and annihilation operators, with others vanishing. It will be useful to define $a(k) \equiv \sqrt{2\omega_k}a_{\vec{k}}$, so then the above becomes

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega(k)}[a(k)e^{-ikx} + a^\dagger(k)e^{ikx}],$$

$$[a(k), a^\dagger(k')] = (2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{k}'),$$

with the relativistic-invariant measures appearing.

The quantum field ϕ is a superposition of creation and annihilation operators. Also, plugging into our expressions for energy and momentum gives the operators

$$H = \tfrac{1}{2}\int \frac{d^3k}{(2\pi)^2(2\omega)}\omega(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})),$$

$$\vec{P} = \tfrac{1}{2}\int \frac{d^3k}{(2\pi)^2(2\omega)}\vec{k}(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})),$$

Need to normal order the first term. Define $: AB :$ for operators A and B to mean that the terms are arranged so that the annihilation operators are on the right, so annihilates the vacuum.

- The vacuum $|0\rangle$ is annihilated by all $a(k)$. Create states with momenta p_1^μ, ..., p_n^μ via $a^\dagger(p_1)\dots a^\dagger(p_n)|0\rangle$. Note that these behave as identical bosons: the state is symmetric under exchanging any pair of momenta, because $[a^\dagger(p), a^\dagger(p')] = 0$.
- Two-point field correlation function:

$$\langle 0|\phi(x)\phi(y)|0\rangle \equiv D(x-y) = \int \frac{d^3k}{(2\pi)^3 2\omega(k)}e^{-ik(x-y)}.$$

Note also that $2i\partial_{x^0}D(x-y)$ is the integral that we saw in last lecture, for the probability amplitude to find a particle having traveled with spacetime displacement $(x-y)^\mu$. For spacelike separation, $(x-y)^2 = -r^2$, we here get $D(x-y) = \frac{m}{2\pi^2 r}K_1(mr)$, with K_1 a Bessel function. Recall that the Bessel function has a simple pole when its argument vanishes, and exponentially decays at infinity. So $D(x-y) \sim \exp(-m|\vec{x}-\vec{y}|)$ is non-vanishing outside the forward light cone.