10/7 Lecture outline

⋆ **Reading for today's lecture: Coleman to end of lecture 4 (p. 37).**

• Last time: write

$$\phi(x) = \phi^+(x) + \phi^-(x),$$

with (backwards looking Heisenberg / Pauli notation)

$$\phi^+(x) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega(k)} a(k) e^{-ikx}, \qquad \phi^-(x) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega(k)} a(k)^\dagger e^{ikx}$$

and $[a(k), a^\dagger(k')] = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}')$ where $\phi^\pm$ are positive / negative frequency. Historically, first attempt was to keep just ϕ_+ and regard it as a quantum wavefunction, ψ, with probability $\sim |\psi|^2$. Doesn't work.

Define normal ordering $: AB :$ for operators A and B: the terms are arranged so that the annihilation operators are on the right, so annihilates the vacuum, e.g. $: \phi^+(x)\phi^-(y) := \phi^-(y)\phi^+(x)$. Take

$$H \equiv : H := \int \frac{d^3k}{(2\pi)^2 (2\omega)} \omega a^\dagger(\vec{k}) a(\vec{k}),$$

$$\vec{P} \equiv : \vec{P} := \int \frac{d^3k}{(2\pi)^2 (2\omega)} \vec{k} a^\dagger(\vec{k}) a(\vec{k}).$$

We're dropping the CC contributing term in H, as discussed last time. So $P^\mu|0\rangle = 0$ and $P^\mu|p_1 \dots p_n\rangle = p^\mu_{tot}|p_1 \dots p_n\rangle$, where $|p_1 \dots p_n\rangle = \prod_n a^\dagger(k_n)|0\rangle$ and $p^\mu_{tot} = \sum_n p^\mu_n$.

• Last time: two-point field correlation function:

$$\langle 0|\phi(x)\phi(y)|0\rangle \equiv D_1(x-y) = \int \frac{d^3k}{(2\pi)^3 2\omega(k)} e^{-ik(x-y)}.$$

Note also that $2i\partial_{x^0} D(x-y)$ is the integral that we saw in last lecture, for the probability amplitude to find a particle having traveled with spacetime displacement $(x-y)^\mu$. For spacelike separation, $(x-y)^2 = -r^2$, we here get $D(x-y) = \frac{m}{2\pi^2 r} K_1(mr)$, with K_1 a Bessel function. Recall that the Bessel function has a simple pole when its argument vanishes, and exponentially decays at infinity. So $D(x-y) \sim \exp(-m|\vec{x} - \vec{y}|)$ is non-vanishing outside the forward light cone. We will soon discuss how to construct physical observables, like S-matrix elements, from squaring amplitudes, and how to construct amplitudes from field correlation functions. For the moment, suffice it to say that the above above correlator is not directly a physical observable, and having it not vanish outside the light cone does not imply any a-causality.

• Causality? There could be observable effects, from interference, if a commutator of fields is non-vanishing outside of the lightcone. Let's show that this does not happen. Note that

$$[\phi(x), \phi(y)] = [\phi^+(x), \phi^-(y)] + [\phi^-(y), \phi^+(x)] =$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega(k)} \int \frac{d^3k'}{(2\pi)^3 2\omega(k')} [a(k), a^\dagger(k')] e^{-ikx+ik'y} - (x \leftrightarrow y)$$

Note that the commutator is a c-number, not an operator:

$$[\phi(x), \phi(y)] = D_1(x-y) - D_1(y-x),$$

where $D_1(x-y)$ is as defined above. For spacelike separation, $(x-y)^2 = -r^2$, $D_1(x-y) = \frac{m}{2\pi^2 r} K_1(mr)$, with K_1 a Bessel function. For spacelike separation, we can map $(x-y)^\mu$ to $-(x-y)^\mu$ by a Lorentz transformation, so $D_1(x-y) - D_1(y-x) = 0$. Good. The commutator is non-vanishing for timelike separation.

Note that $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$, wouldn't have been true for just $\phi^+(x)$, so there would be information propagating outside the light cone. Moreover, neither $|\phi|^2$ nor $|\phi^+|^2$ can be interpreted as a conserved probability – the relativistic expression $E = \sqrt{\vec{p}^2 + m^2}$ necessarily leads to particle productions. So instead we interpret ϕ as similar to $\vec{x}$ in QM, as a hermitian operator, not a wavefunction.

• Get more interesting theories by adding interactions, e.g. $V(\phi) = \frac{1}{2}m^2\phi^2 + \lambda\phi^4$, treat 2nd term as a perturbation. We can consider perturbative solutions in both classical or quantum field theory. The starting point is the green's function for the theory with a forcing function source:

• Consider $\mathcal{L} = \frac{1}{2}\partial\phi^2 - \frac{1}{2}m^2\phi^2 - \rho\phi$, where ρ is a classical source. Solve by $\phi = \phi_0 + i\int d^4y D(x-y)\phi(y)$, where ϕ_0 is a solution of the homogeneous KG equation and the green's function $D(x-y)$ satisfies

$$(\partial_x^2 + m^2) D(x-y) = -i\delta^4(x-y).$$

By a F.T., get

$$D_?(x-y) = \int_? \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} e^{-ik(x-y)}.$$

Consider the k_0 integral in the complex plane. There are poles at $k_0 = \pm\omega_k$, where $\omega_k \equiv +\sqrt{\vec{k}^2 + m^2}$. There are choices about whether the contour goes above or below the poles, and that's what the ? label indicates.

• Going above both poles gives the retarded green's function, $D_R(x-y)$ which vanishes for $x_0 < y_0$. Considering $x_0 > y_0$, get that

$$D_R(x-y) = \theta(x_0 - y_0)\int \frac{d^3k}{(2\pi)^3 2\omega_k}(e^{-ik(x-y)} - e^{ik(x-y)})$$
$$\equiv \theta(x_0 - y_0)(D_1(x-y) - D_1(y-x)) = \theta(x_0 - y_0)\langle[\phi(x), \phi(y)]\rangle,$$

where $D_1(x-y)$ is as defined above. This is reasonable: the $\rho(y)$ source only affects $\phi(x)$ in the future.

Going below both poles gives the advanced propagator, which vanishes for $y_0 < x_0$.

• Feynman propagator: go above the $k_0 = E_k$ pole and below the $k_0 = -E_k$ pole. $-E_k$ pole is heuristically the anti-matter, traveling backward in time. Show that this gives

$$D_F = \theta(x_0 - y_0)D_1(x-y) + \theta(y_0 - x_0)D_1(y-x).$$

Now show

$$D_F(x-y) \equiv \langle T\phi(x)\phi(y)\rangle = \begin{cases} \langle\phi(x)\phi(y)\rangle & \text{if } x_0 > y_0 \\ \langle\phi(y)\phi(x)\rangle & \text{if } y_0 > x_0 \end{cases}.$$

Here T means to time order: order operators so that earliest is on the right, to latest on left. Object like $\langle T\phi(x_1)\dots\phi(x_n)\rangle$ will play a central role in this class. Time ordering convention can be understood by considering time evolution in $\langle t_f|t_i\rangle$. Evaluate $D_F(x-y)$ by going to momentum space:

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon}e^{-ik(x-y)},$$

where $\epsilon \to 0^+$ enforces that we go below the $-\omega_k$ pole and above the $+\omega_k$ pole, i.e. we get $D(x-y)$ if $x_0 > y_0$, and $D(y-x)$ if $x_0 < y_0$, as desired from the definition of time ordering. We'll see that this ensures causality.

• The pole placement is such that the contour can be rotated to be along the imaginary k_0 axis, running from $-i\infty$ to $+i\infty$. This will later tie in with a useful way to treat QFT, by going to Euclidean space via imaginary time. It is something of a technical trick, but there is also something deep about it. Analyticity properties of amplitudes is deeply connected with causality. More later.

• Define contraction of two fields $A(x)$ and $B(y)$ by $T(A(x)B(y)) - :A(x)B(y):$. This is a number, not an operator. Let e.g. $\phi(x) = \phi^+(x) + \phi^-(x)$, where ϕ^+ is the term with annihilation operators and ϕ^- is the one with creation operators (using Heisenberg and Pauli's reversed-looking notation). Then for $x^0 > y^0$ the contraction is $[A^+, B^-]$, and for $y^0 > x^0$ it is $[B^+, A^-]$. So can put between vacuum states to get that the contraction is $\langle TA(x)B(y)\rangle$. For example, in the KG theory the contraction of $\phi(x)$ and $\phi(y)$ is $D_F(x-y)$.