11/21/16 Lecture 16 outline

• Aside: $Y_{\ell,m}(\theta,\phi)$ asides: $Y_{\ell,\ell}=c_\ell e^{i\ell\phi}\sin^\ell\theta$, $Y_{\ell,0}=\sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta)$,

$$Y_{\ell,m}=c_{\ell,m}e^{im\phi}(\sin\theta)^{-m}\frac{d^{\ell-m}}{d(\cos\theta)^{\ell+m}}(\sin\theta)^{2\ell}.$$

In rectangular coordinates, the $Y_{\ell m}$ are given by appropriate $F_\ell(x,y,z)/r^\ell$ which we can understand in terms of addition of angular momentum. E.g. $r^2=x^2+y^2+z^2$ has $\ell=m=0$ whereas $Q_{ij}=(x_ix_j-\frac{1}{3}r^2)/r^2$ has $\ell=2$, with the five independent components corresponding to $m=2,1,0,-1,-2$. This is referred to as an $\ell=2$ tensor. Likewise can consider $Q_{i_1,\dots i_\ell}$ by symmetrizing and subtracting the traces.

• Last time: spherically symmetric $V(r)$ means that $[H,L_a]=0$, so we can find simultaneous eigenstates $|E,\ell,m\rangle$. Writing $\vec{p}^2\to p_r^2+L^2/r^2$, we find that $\vec{x}|E\ell m\rangle=R_{E,\ell}(r)Y_{\ell,m}(\theta,\phi)$, where $R_{E,\ell}$ satisfies the radial SE

$$\left(-\frac{\hbar^2}{2mr^2}\frac{d}{dr}(r^2\frac{d}{dr})+\frac{\ell(\ell+1)\hbar^2}{2mr^2}+V(r)\right)R_{E,\ell}(r)=ER_{E,\ell}(r).$$

It looks a little nicer for $u_{E,\ell}(r)\equiv rR_{E,\ell}(r)$:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+V_{eff}(r)u=Eu,\qquad V_{eff}=V(r)+\frac{\ell(\ell+1)\hbar^2}{2mr^2}.$$

If we assume that $r^2V(r)\to 0$ for $r\to 0$, then the angular momentum barrier wins and the SE implies $u(r)\to Ar^{\ell+1}+Br^{-\ell}$ in this limit, and the condition that $j_r=\hat{r}\cdot\vec{j}=\frac{\hbar}{m}Im(\psi^*\partial_r\psi)\to 0$ for $r\to 0$ excludes the second term, so $R_{E,\ell}\to r^\ell$ for $r\to 0$. So wavefunction vanishes at origin except for $\ell=0$; this is the angular momentum barrier.

• Free particle in spherical coordinates: $\psi_{E,\ell,m}=R_{E,\ell}(r)Y_{\ell,m}(\theta\phi)$ has $R=u/r$ with

$$(\frac{d^2}{dr^2}+k^2-\frac{\ell(\ell+1)}{r^2})u=0,\qquad \hbar k=\sqrt{2mE}.$$

Function of $\rho=kr$. Solutions of the ODE in ρ are the spherical Bessel functions

$$j_\ell=(-\rho)^\ell(\frac{1}{\rho}\frac{d}{d\rho})^\ell(\frac{\sin\rho}{\rho}).$$

(Replacing $\sin\rho-\to-\cos\rho$ gives the spherical Neumann functions $n_\ell(\rho)$ which also solve the ODE but have $n_\ell\sim\rho^{-\ell-1}$ for $\rho\to 0$ which is badly behaved and thus thrown away. As expected, here k and E are continuous. The solutions are delta-function normalizable:

$$\int_0^\infty j_\ell(kr)j_{\ell'}(k'r)r^2dr=\frac{\pi}{2k^2}\delta(k-k')\delta_{\ell\ell'}.$$

• Particle in a spherical well of radius a: need to impose $j_\ell(ka) = 0$, leads to quantized $k \to k_{\ell,n}$ e.g. $k_{0,n} = n\pi$, and thus $E_{n,\ell}$. No degeneracy in ℓ.

• SHO: $r \equiv \sqrt{\hbar/m\omega}\rho$, $u = \rho^{\ell+1}e^{-\rho^2/2}f(\rho)$ gives an eqn for $f(\rho) \equiv \sum_{n=0}^{\infty} a_n \rho^n$ with recursion relation

$$a_{n+2} = \frac{2n + 2\ell + 3 - 2E/\hbar\omega}{(n+2)(n+2\ell+3)} a_n.$$

For $n \to \infty$ this recursion relation gives $f \sim e^{\rho^2}$, which would lead to a non-normalizable ψ, so there has to be some $n = q$ where it truncates, i.e. $a_{n>q} = 0$. This leads to $E = (2q + \ell + \frac{3}{2})\hbar\omega$, where $q = 0, 1, 2, \ldots$ is the number of nodes in the radial wavefunction. Compare to rectangular coordinates and 3 decoupled SHOs, where we get $E = (N + \frac{3}{2})\hbar\omega$, get $N = n_1 + n_2 + n_3 = 2q + \ell$. Note degeneracy with different ℓ having same E.

• Coulomb potential: $V = -Ze^2/r$. Usual to write in terms of $\alpha = e^2/\hbar c \approx 1/137$. Coulomb potential: define $\rho \equiv \kappa r$ where $\hbar\kappa \equiv \sqrt{2m|E|}$ and $\rho_0 \equiv \sqrt{2m/|E|}(Ze^2/\hbar)$ and use $\alpha \equiv e^2/\hbar c \approx 1/137$. Then $u_{E,\ell} \equiv \rho^{\ell+1}e^{-\rho}w(\rho)$ solves the radial S.E. if $w(\rho)$ satisfies an ODE. The solutions can be written in terms of hypergeometric functions. As usual for bound state problems, we find that E has to be quantized or the solution would be badly behaved for $r \to \infty$, would get $w(\rho) \to e^\rho$ for generic E. To avoid this, the series for $w(\rho)$ must truncate at finite order N. This requires $\rho_0 = 2(N + \ell + 1)$. Note degeneracy.

Upshot: Find that the radial equation gives $E_n = -\frac{1}{2}mc^2 Z^2 \alpha^2/n^2$ where $n = N + \ell + 1$, with $N = 0, 1, 2 \ldots$, i.e. $\ell = 0, 1, \ldots, n-1$. The degeneracy for fixed n is $\sum_{\ell=0}^{n-1}(2\ell+1) = n^2$. Taking $a_0 \equiv \hbar^2/me^2$, (with F a Hypergometric function)

$$R_{n,\ell}(r) \propto r^\ell e^{-Zr/na_0} F(-n + \ell + 1; 2\ell + 2; 2Zr/na_0).$$