10/12/16 Lecture 6 outline

• Momentum generates spatial translations, and the Hamiltonian (energy) generates time translations. Schrodinger equation: $i\hbar\partial_t U(t,t_0) = HU(t,t_0)$. For time independent H, this is simple to integrate: $U = e^{-iH(t-t_0)/\hbar}$. For t dependent H, one gets instead (Dyson) $U(t,t_0) = T\exp(-(i/\hbar)\int_{t_0}^t dt' H(t'))$, where T denotes time ordering. We won't need that here. Since time evolution is a unitary operator, it preserves the norm of kets.

• Schrodinger picture has the time dependence in the bras and kets: $|\psi(t)\rangle_S = U(t,t_0)|\psi(t_0)\rangle$. So $|\psi(t)\rangle_S$ satisfies the Schrodinger equation: $i\hbar\partial_t|\psi(t)\rangle_S = H|\psi(t)\rangle_S$. The Heisenberg picture is equivalent, but instead puts the t dependence in the operators:

$$\langle\mathcal{O}\rangle = {}_S\langle\psi(t)|\mathcal{O}_S|\psi(t)\rangle_S = \langle\psi(t_0)|U(t,t_0)^\dagger\mathcal{O}U(t,t_0)|\psi(t_0)\rangle \equiv \langle\psi(t_0)|\mathcal{O}^H(t)|\psi(t_0)\rangle,$$

where $\mathcal{O}^H = U^\dagger\mathcal{O}U$ is the Heisenberg picture operator. Note that it satisfies

$$\frac{d}{dt}\mathcal{O}^H = \frac{1}{i\hbar}[\mathcal{O}^H, H] + \frac{\partial}{\partial t}\mathcal{O}^H.$$

This is exactly what one finds in classical mechanics, in terms of Poisson brackets, with Dirac's replacement $[A,B]_{QM} = i\hbar[A,B]_{PB}$. For example, if $H = \vec{p}^2/2m + V(\vec{x})$, the above gives

$$\frac{d}{dt}\vec{x}^H(t) = \vec{p}^H(t)/m, \qquad \frac{d}{dt}\vec{p}^H(t) = -\partial_{\vec{x}}V(\vec{x}).$$

These look like the classical equations of motion in the Hamiltonian formulation; they apply as operator statements to the Heisenberg picture operators.

• Free particle: Heisenberg picture operators have $\vec{p}(t) = \vec{p}(0) = \vec{p}$ and $\vec{x}(t) = \vec{x}_0 + \vec{p}t/m$; these are all operators. So $[x_i(t), x_i(0)] = -i\hbar t/m$, and thus $\langle(\Delta x_i(t))^2\rangle\langle(\Delta x_i(0)^2\rangle \geq \hbar^2 t^2/4m^2$.

• E.g. if there is an external magnetic field in between the SG experiments, it'll make the spin of the state precess via $H = -\vec{\mu}\cdot\vec{B}$ with $\vec{\mu} = ge\vec{S}/2mc$. Take e.g. $\vec{B} = B_0\hat{z}$ so $H = g|e|S_z B/2mc \equiv \omega S_z$ Then $U = e^{-iHt/\hbar}$ is diagonal in the $|\pm_z\rangle$ basis. Show e.g. that if in the $|+_x\rangle$ state at $t_0 = 0$, the probability of finding it later in the $|\pm_x\rangle$ state is $\cos^2(\omega t/2)$ and $\sin^2(\omega t/2)$, respectively, and $\langle S_x\rangle = \frac{1}{2}\hbar\cos\omega t$ and $\langle S_y\rangle = \frac{1}{2}\hbar\sin\omega t$ and $\langle S_z\rangle = 0$, fitting with the classical picture of precessing in the xy plane with frequency ω. Recall from HW that $e^{i\theta\hat{n}\cdot\vec{\sigma}} = \cos\theta\mathbf{1} + i\sin\theta\hat{n}\cdot\vec{\sigma}$. Work out $S_x(t)$ in that basis.