Physics 105a, Ken Intriligator lecture 16, Nov 28, 2017

• Continue with chapter 3. Introducing partial differential equations, and their solution by separation of variables. First example, the string wave equation:

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2})y(x,t) = 0,$$

where $c = \sqrt{\tau/\mu}$ is the wave velocity on the string and $y(x,t)$ is the string's displacement from equilibrium. Aside: This equation comes from $F_y = ma_y$ for the bit of string between x and $x+dx$: write it as $dF_y = dma_y$ where the d is because it is a small element e.g. its mass is $dm = \mu dx$ and $dF_y = F_y(x+dx) - F_y(x) = \partial x F_y dx$ and $F_y(x) = T\partial y/\partial x$. Aside: the string wave equation can be obtained from the Euler Lagrange equations with

$$S = \int dt \int dx (\tfrac{1}{2}\mu\dot{y}^2 - \tfrac{1}{2}Ty'^2).$$

where $\dot{f} \equiv \partial_t f$ and $f' \equiv \partial_x f$.

The string wave equation is can be called the wave equation in 1+1 dimensions. There is a similar equation in 2+1 dimensions, describing for example waves on a 2d surface (like a drum head), or in 3+1 dimensions

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\psi(x,t) = 0$$

where ψ could be the longitudinal displacement of air in a sound wave, or it could be a transverse component of $\vec{E}$ or $\vec{B}$ in a light wave.

• Note that the wave equations are linear, so they can be solved by superposition. One class of solutions are left and right moving traveling waves

$$y(x,t) = f(x+ct) + g(x-ct)$$

which satisfies the wave equation for any functions f and g. For fixed ends (Dirichlet boundary conditions), a traveling wave that hits the end gets reflected back and satisfies $y(x = x_{end}, t) = 0$ by having the reflected wave get flipped upside down. (You probably saw essentially this same thing in a class on waves in the context of the phase shift when light reflects off of a medium with larger index of refraction.)

• Illustrating separation of variables for the string wave equation. Take $y = f(t)\psi(x)$ and plug into the equation. Put all of the t-dependent terms on one side of the equation, and all of the x-dependent terms on the other. The equation has to be satisfied for all (t,x).

This can only work if both sides are a constant. The upshot is two ordinary differential equations. For the case of the string, both of these equations are simply that of a SHO:

$$\frac{d^2 f}{dt^2} = -\omega^2 f, \qquad \frac{d^2\psi(x)}{dx^2} = -k^2\psi(x), \qquad \omega = ck.$$

Suppose that there are Dirichlet boundary conditions with $\psi(x=0) = \psi(x=L) = 0$. Then the solutions for ψ are labeled by integers (there is no solution unless $k = k_n$ takes special values):

$$\psi_n(x) = C_n \sin(n\pi x/L), \qquad \text{i.e.} \quad k_n = 2\pi/\lambda_n = n\pi/L \qquad n = 1, 2, 3, \ldots$$

where $n = 1$ is the fundamental mode, $n = 2$ is the first excited mode (which has one node in the middle, $n = 3$ is the second excited mode, etc., $\lambda_n = 2L/n$, and C_n are constants. These are the same as the solutions for the wave function of a particle in a 1d box in quantum mechanics, where there $p_n = \hbar k_n$ (because ψ satisfies the same differential equation and the same boundary conditions). Since $k = k_n$, the time-dependent equation has frequency $\omega = \omega_n = ck_n$. We can get a general solution by superposition

$$y(x,t) = \sum_{n=1}^{\infty} (A_n \cos(cn\pi t/L) + B_n \sin(cn\pi t/L)) \sin(n\pi x/L).$$

The A_n and B_n are constants of integration, that can be determined from the initial conditions. If we are told that $y(x,0) = y_0(x)$ and $\dot{y}(x,0) = v_0(x)$, with $y_0(x)$ and $v_0(x)$ some given functions, then we can related A_n and B_n to the Fourier coefficients in the Fourier series expansion of these functions:

$$A_n = \frac{2}{L}\int_0^L y_0(x) \sin(n\pi x/L) dx, \qquad B_n = \frac{2}{L\omega_n}\int_0^L v_0(x) \sin(n\pi x/L) dx.$$

• Dirichlet boundary conditions: $y(0,t) = y_0$, $y(L,t) = y_L$, fixed. Neumann boundary conditions: instead specify the slope at the ends, $y_x(0,t)$ or $y_x(L,t)$. Can have NN, ND, DN, DD boundary conditions (i.e. either one, at either end), depending on the physical setup. For D conditions at $x = 0$ and N at $x = L$, get $\psi_n = C_n \sin(k_n x)$ with $k_n L = (n + \frac{1}{2})\pi$. For N conditions at $x = 0$, use instead $\psi_n(x) = C_n \cos(k_n x)$ and then determine k_n by conditions at $x = L$.

• Example: take $y_0(x)$ to be a straight line, of slope a for $0 < x < L/2$ and slope $-a$ for $L/2 < x < L$, with $v_0(x) = 0$. So $B_n = 0$ and $A_n = 4aL\sin(n\pi/2)/n^2\pi^2$. Show animation from Dubin cell 3.3 for the first $M = 30$ terms. Looks strange, but it's correct for undamped oscillations.

• Example: Gaussian $y_0(x) = e^{-50(x-L/2)^2/L^2}$, $v_0(x) = 0$. Again, animations from Dubin.

• Heat flow equation in $1+1d$: $C(x)\partial_t T = \partial_x(\kappa \partial_x T) + S$, where C is the specific heat, κ is the thermal conductivity, and S is the source of heat energy per unit volume. For κ and C constant, this becomes $\partial_t T = \chi \partial_x^2 T + S/C$, where $\chi = \kappa/C$ is the thermal diffusivity (units of m^2/s e.g. $\chi \sim 10^{-7}$ for water and $\chi \sim 10^{-4}$ for copper. Initial condition in time $= T(x,0) = T_0(x)$. Initial condition in space e.g. D boundary conditions: $T_1(t)$ and $T_2(t)$ at the two ends (or N conditions specifying T_x at either or both ends.

Separation of variables works if there is an equilibrium solution $T_{eq}(x)$, that is time independent. Necessary to have t independent boundary conditions and source. The solution then takes the form $T(x,t) = T_{eq}(x) + \tilde{T}(x,t)$ (where $\tilde{T} \equiv \Delta T$ in the book). Then $\partial_t \tilde{T} = \chi \partial_x^2 \tilde{T}$. As usual, separate variables, taking $\tilde{T} = f(t)\psi(x)$ to get $\partial_t \ln f = \lambda$ and $\chi \partial_x^2 \psi = \lambda \psi$, with λ a constant. Typically the setup gives $\lambda < 0$ to avoid exponentially growing solutions in time and space, so get e.g. (in the DD boundary condition case; for NN get instead $\cos(n\pi x/L)$ solutions).

$$T(x,t) = T_{eq}(x,t) + \sum_n A_n e^{-\chi t (n\pi/L)^2} \sin(n\pi x/L).$$

• Now consider Laplace and Poisson's equations, $\nabla^2 \phi = -\rho$, Or the wave equation in more space dimensions, $\nabla^2 \psi - \frac{1}{c^2}\partial_t^2 \psi = 0$. Or the heat equation in more space dimensions, $\partial_t T = \chi \nabla^2 T$. In all of them we replace the 1d ∂_x^2 with ∇^2. The reason for this is rotational symmetry. Separation of variables is then to take $\psi = f(t)X(x)Y(y)Z(z)$ if the setup is rectangular. Or $\psi = f(t)R(r)\Theta(\theta)\Phi(\phi)$ if the setup is spherical. Or $\psi = f(t)P(\rho)\Phi(\phi)Z(z)$ if it's cylindrical. If it's none of these, there are some other separable coordinate systems. If it's none of those, it might be better to just give it to a computer.

• Solutions of the Laplace equation with N or D boundary conditions are unique: if $\Phi = \phi_1 - \phi_2$ is the difference then $0 = \int_V \Phi \nabla^2 \Phi dV = \int_{\partial V} \Phi \nabla \Phi \cdot d\vec{a} - \int_V \nabla \Phi \cdot \nabla \Phi$, and then the BCs ensure that the $\int_{\partial V}$ term $= 0$, and positivity $\nabla \Phi \cdot \nabla \Phi$ implies that this integral can only vanish if Φ is a constant, which must be zero to satisfy the BCs.

• E.g. 2d rectangular: $\phi(x,y) = X(x)Y(y)$ with $X(x) = C_1 e^{\kappa x} + C_2 e^{-\kappa x}$ and $Y(y) = C_3 e^{i\kappa y} + C_4 e^{-i\kappa y}$, where κ is real or imaginary depending on the BCs.

• 2d cylindrical: $\phi(r,\theta) = R(r)\Theta(\theta)$, then $\nabla^2\Phi/\Phi = (rR)^{-1}\partial_r(r\partial_r R) + (r^2\Theta)^{-1}\partial_\theta^2\Theta$. Solutions of $\nabla^2\phi = 0$ are $\phi = A_0 + B_0 \ln r + \sum_{m\neq 0}(A_m r^{|m|} + B_m r^{-|m|})e^{im\theta}$.

• Spherical: $\nabla^2\Psi = r^{-2}\partial_r(r^2\partial_r\Psi) + (r^2\sin\theta)^{-1}\partial_\theta(\sin\theta\partial_\theta\Psi) + (r^2\sin^2\theta)^{-1}\partial_\phi^2\Psi$. Solutions of $\nabla^2\Psi = 0$ are found by taking $\Psi = R(r)\Theta(\theta)\Phi(\phi)$ find $\Phi = e^{im\phi}$ and $\Theta(\theta) = P_\ell^m(\cos\theta)$ the associated Legendre functions. Finally, $R_{\ell,m}(r) = A_{\ell,m}r^{-\ell-1} + B_{\ell,m}r^\ell$. So $\Psi = \sum_{\ell=0}^\infty\sum_{m=-\ell}^{\ell}(A_{\ell,m}r^{-\ell-1} + B_{\ell,m}r^\ell)e^{im\phi}P_\ell^m(\cos\theta)$. Use $Y_{\ell,m} = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\phi}$ as the orthonomal, complete basis of functions of (θ,ϕ).