Physics 105a, Ken Intriligator lecture 19, Dec 7, 2017

• Several physical situations that involve essentially the same mathematics: (i) Laplace equation, (ii) heat equation, (iii) wave equation, (iv) Schrodinger equation (for a free particle in a confining potential box). In all of them we solve the following boundary value problem in space: find the eigenstates and eigenvalues of the Laplacian in the appropriate geometry, and write the solution as a superposition of them, with coefficients determined by the boundary conditions on the geometry's boundary.

For the wave equation $(\partial_t^2 - c^2\nabla^2)\psi = 0$. For the heat equation $(\partial_t - \chi\nabla^2)\tilde{T} = 0$. For the Schrodinger equation $(i\hbar\partial_t + \frac{\hbar^2}{2m}\nabla^2 - V)\psi = 0$. For all of them the first step is to solve for the eigenfunctions and eigenvalues of the Laplacian:

$$\nabla^2\psi_{\vec{n}}(\vec{x}) = -\lambda_{\vec{n}}\psi_{\vec{n}}(\vec{x}).$$

We can regard the Laplace equation as a special case, with $\lambda = 0$, of this more general equation. The minus sign is because the physical boundary conditions and behavior favor such solutions, with $\lambda > 0$, which are oscillating, compared with the exponential solutions. In 1d, we could either solve $\psi'' = +\kappa^2\psi$ or $\psi'' = -k^2\psi$, and the former has exponential solutions whereas the latter has oscillating, and we're aiming for oscillating.

The $\vec{n}$ labels the solutions. E.g. for $\psi_n'' = -k_n^2\psi_n$ with D boundary conditions on the two ends we get $\psi_n = \sin(n\pi x/L)$ and $k_n = -n^2\pi^2/L^2$. In d dimensions the similar equation has d labels, so $\vec{n}$ denotes 3 labels in 3d. For example, in a 3d cube, with sides of length a, b, and c, and D boundary conditions on each edge, we have

$$\psi_{n_1,n_2,n_3}(\vec{x}) = \sin(n_1\pi x/a)\sin(n_2\pi y/b)\sin(n_3\pi z/c), \qquad \lambda_{\vec{n}} = -\pi^2((n_1/a)^2+(n_2/b)^2+(n_3/c)^2).$$

The $\psi_{\vec{n}}(x)$ form a complete basis for functions, so now one considers e.g.

$$\psi(\vec{x}) = \sum_{\vec{n}} A_{\vec{n}}\psi_{\vec{n}}(\vec{x}),$$

where the $A_{\vec{n}}$ are determined by the initial conditions and/or the boundary conditions on the boundary of the region. For the wave equation and heat equation one obtains

$$\psi(t,\vec{x}) = \sum_{\vec{n}}(A_{\vec{n}}\cos\omega_{\vec{n}}t + B_{\vec{n}}\sin(\omega_{\vec{n}}t))\psi_{\vec{n}}(\vec{x}), \qquad \tilde{T} = A_n\sum_{\vec{n}} e^{-\chi\lambda_{\vec{n}}t}\psi_{\vec{n}}(\vec{r}).$$

where $\omega_{\vec{n}} \equiv c\sqrt{\lambda_{\vec{n}}}$.

• E.g. heat equation on a 2d rectangle with $S = 0$. Then $\nabla^2 T_{eq}(x, y) = 0$ requires solving the Laplace equation with appropriate BCs. The solution for $\tilde{T}(x, y, t)$ is obtained by separation of variables. For example, with Dirichlet BCs at the ends get

$$\tilde{T} = \sum_{n>0} \sum_{m>0} A_{n,m} e^{-\chi\pi^2(n^2/a^2+m^2/b^2)t} \sin(n\pi x/a) \sin(m\pi y/b)$$

where $A_{n,m}$ is obtained from the initial conditions as

$$A_{n,m} = \frac{4}{ab} \int_0^a dx \int_0^b dy \sin(n\pi x/a) \sin(n\pi y/b)(T_0(x, y) - T_{eq}(x, y)).$$

• In cylindrical coordinates (ρ, ϕ, z) and spherical coordinates (r, θ, ϕ) we have

$$\nabla^2 \psi = \frac{1}{\rho} \partial_\rho(\rho \partial_\rho \psi) + \frac{1}{\rho^2} \partial_\phi^2 \psi + \partial_z^2 \psi = \frac{1}{r} \partial_r^2(r\psi) - \frac{1}{r^2} L^2 \psi,$$

where, to save space (and because it is related to angular momentum) I have defined

$$-\widehat{L}^2 \psi \equiv \frac{1}{\sin\theta} \partial_\theta(\sin\theta \partial_\phi \psi) + \frac{1}{\sin^2\theta} \partial_\phi^2 \psi.$$

Consider first the ϕ derivatives, we have

$$-\partial_\phi^2 \Phi(\phi) = -m^2 \Phi(\phi), \qquad \Phi(\phi) = e^{im\phi}.$$

For cylindrical coordinates separation of variables gives

$$\partial_z^2 Z(z) = -k_z^2 Z(z),$$

where we will either take k_z real or k_z imaginary depending on the setup. For example, consider $\nabla^2 \phi = 0$. Then, taking $\phi = R(\rho)\Phi(\phi)Z(z)$ we see that the ∂_ϕ^2 term is negative so to get $\nabla^2 \phi = 0$ we need some positive contributions, i.e. we need exponential behavior in either the z or the ρ direction. Which one is determined by the BCs, either we'll need oscillating solutions in z or oscillating in ρ, and then the other must be exponential.

First recall 2d from last time: the solutions are $\phi = A_0 + B_0 \ln r + \sum_{m\neq 0}(A_m r^{|m|} + B_m r^{-|m|})e^{im\theta}$. E.g. suppose that there is a cylinder of radius a and the potential at $r = a$ is $V_a(\theta)$. The solution for $r < a$ has $B_m = 0$ and the solution for $r > a$ has $A_m = 0$. The solution on the boundary has e.g. $A_m a^{|m|} = \oint d\theta V_a(\theta) e^{-im\theta}/2\pi$, i.e. the familiar Fourier transform expressions.

For 3d cylindrical, we have $\phi(r,\theta,z) = R(r)\Theta(\theta)Z(z)$ with $Z'' = k^2 Z$ with, taking $u \equiv kr$ the equation for $R(u)$ is the Bessel equation: $R'' + u^{-1}R' + (1 - m^2 u^{-2})R = 0$. E.g. $\phi(r,\theta,z=0) = 0$, $\phi(r,\theta,z=L) = V_0(r,\theta)$ and $\phi(a,\theta,z) = 0$ has solution $\Theta(\theta) = e^{im\theta}$ and $Z(z) = \sinh(k_{mn}z)$ and $R_{m,n}(r) = AJ_m(k_{mn}r) + BN_m(k_{mn}r)$ where $B = 0$ for the solution be be finite at $r = 0$ and $k_{mn} = j_{mn}/a$ and $j_{mn} = BesseJZero[m,n]$.

$$\phi(r,\theta,z) = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} A_{m,n}e^{im\phi}J_m(k_{n,m}r)\sinh(k_{n,m}z).$$

where we get $A_{n,m}$ by inverting the requirement that $\phi(r,\theta,L) = V_0(r,\theta)$. This is done by using orthogonality properties of the Bessel functions:

$$\int_0^a rJ_n(x_{n,m}r/a)J_n(x_{n,m'}r/a)dr = \tfrac{1}{2}a^2 J_{n+1}(x_{n,m})^2\delta_{m,m'}.$$

• Spherical: take $\psi_{n,\ell,m}(r,\theta,\phi) = R_{n,\ell}(r)Y_{\ell,m}(\theta,\phi)$, where the $Y_{\ell,m}$ are eigenvalues of $\widehat{L}^2$ defined above:

$$-\widehat{L}^2 Y_{\ell,m} = -\ell(\ell+1)Y_{\ell,m}(\theta,\phi),$$

with $\ell = 0,1,2\ldots$. This might look familiar from QM, and you will see a lot more of it in the upper division QM class. The $Y_{\ell,m}$ form a complete basis for functions of θ and ϕ with the usual periodicity. Expanding functions in terms of them is conceptually similar to a Fourier transform, but in both θ and ϕ. They are given by $Y_{\ell,m} = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\phi}$ where $P_\ell^m(\cos\theta)$ the associated Legendre functions. Mathematica: $P_\ell^m(x) = LegendreP[l,m,x]$.

Separation of variables then gives

$$\nabla^2\psi = -\lambda\psi, \qquad \psi = R_{n,\ell}(r)Y_{\ell,m}(\theta,\phi), \qquad r^{-1}(rR)'' - \frac{\ell(\ell+1)R}{r^2} = -\lambda R.$$

For the Laplace equation, $\lambda = 0$, the solutions of the radial equation are $R_{\ell,m}(r) = A_{\ell,m}r^{-\ell-1} + B_{\ell,m}r^{\ell}$. Example: find the potential outside of a sphere of radius a with $V(a,\theta,\phi) = V_0(\Omega)$. Then $\phi = \sum_{\ell,m} A_{\ell,m}r^{-\ell+1}Y_{\ell,m}$ where $A_{\ell,m}a^{-\ell+1} = \int d\Omega Y^*_{\ell,m}(\Omega)V_0(\Omega)$. E.g. for $V_0(\Omega) = V_0 H(\pi/2 - \theta)$ get $A_{\ell,m\neq 0} = 0$, and can do needed integral via Mathematica.

For the 3d wave equation, there is a triple sum, e.g. n,ℓ,m, where ℓ,m are the usual $Y_{\ell,m}$ labels, and n comes from the radial solution. Solutions of the Schrodinger equation in QM have the same labels, for the same reason, e.g. the solutions of the Hydrogen atom have ℓ,m giving the angular momentum eigenvalues.

• Disk drum example: $\psi(a,\theta,t) = 0$ with $\psi(r,\theta,0) = z_0(r,\theta)$ and $\partial_t\psi(r,\theta,0) = v_0(r,\theta)$. Separate variables as $\psi = f(t)R(r)e^{im\theta}$ and then $\partial_t^2 f = -\omega_{m,n}^2 f$ where $\omega_{m,n}$ are found from $r^{-1}\partial_r(r\partial_r R) - r^{-2}m^2 R = -\omega_{m,n}^2 R/c^2$, which is Bessel's equation, with $R_{m,n}(r) = AJ_m(\omega_{m,n}r/c) + BY_m(\omega_{m,n}r/c)$, with $B = 0$ to have non-singular behavior at $r = 0$ (or $A = 0$ for non-singular at $r \to \infty$). The n index labels the locations of the zeros of the Bessel's equations solutions, $J_m(j_{m,n}) = 0$.

Traveling wave solutions, e.g. $A_{m,n}e^{i(m\theta-\omega_{m,n}t)}J_m(j_{m,n}r/a)$.

• Oscillations of the surface of a sphere: $(\partial_t^2 - c^2\,\nabla^2)\psi = 0$ in spherical coordinates, taking $r = R$ constant:

$$\psi(t,\Omega) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(A_{\ell,m}\cos\omega_\ell t + B_{\ell m}\sin\omega_\ell t)Y_{\ell,m}(\Omega)$$

with $\omega_\ell = c\sqrt{\ell(\ell+1)}/R$.

• Wave equation in spherical coordinates: get

$$\psi = \sum_n\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(A_{\ell m}\cos\omega_{\ell mn}t + B_{\ell m}\sin\omega_{\ell mn}t)R_{\ell n}(r)Y_{\ell,m}(\theta,\phi).$$

where $R'' + 2r^{-1}R' + (k^2 - \ell(\ell+1)r^{-2})R = 0$ is related to the spherical Bessel equation and $k = \omega/c$. The solutions are $R_\ell = j_\ell(kr) + n_\ell(kr)$, where j_ℓ is the solution that works for $r \to 0$. The integer n is determined by some boundary conditions in r, e.g. for Dirichlet boundary conditions it labels the zeros of j_ℓ. E.g. $j_0(x) = \sin x/x$, $j_1(x) = \sin x/x^2 - \cos x/x$, $n_0 = -\cos x/x$, etc.

• Heat or wave equation in cylindrical coordinates: $\nabla^2\psi_{n,m,k} = -\lambda_{n,m,k}\psi_{n,m,k}$ with e.g. $\psi_{n,m,k} = J_m(j_{m,n}r/a)e^{im\phi}\sin(k\pi z/L)$ and $\lambda_{n,m,k} = j_{n,m}/a + (k\pi/L)^2$.