Physics 105a, Ken Intriligator lecture 8, October 24, 2017

• Fourier transforms, introduction. We will write fairly general functions $f(t)$ as superpositions of terms like $e^{-i\omega t}$, with coefficients that depend on ω. These superpositions can have a finite number of terms, or an infinite number of terms. An example with an infinite number of terms would be a square wave, which requires the infinite number because of the sharp edges of the square, but these terms have smaller and smaller coefficient. The infinite number of terms case can have either summing over discrete values of ω, or integrating over continuous values of ω. The discrete value case occurs for period functions (like the square wave) while the integral version is for more general functions.

• Consider a periodic function $f(t+T) = f(t)$. Examples are $e^{-i\omega_n t}$ with $\omega_n = n\omega_1$, and $\omega_1 \equiv 2\pi/T$. The statement of Fourier transforms is that these functions form a basis for periodic functions. Just as we can expand a vector in terms of basis vectors, we can expand periodic functions in terms of these basis vectors.

$$f(t) = \sum_{n=-\infty}^{\infty} \tilde{f}_n e^{-2\pi i n t/T} = a_0 + \sum_{n=1}^{\infty}(a_n \cos(2\pi n t/T) + b_n \sin(2\pi n t/T)).$$

We can find the f_n coefficients, or the a_n and b_b coefficients (they are related to each other by linear combinations) by using the fact that these basis functions are orthogonal in the following sense:

$$\int_{t_0}^{T+t_0} dte^{-2\pi i(n-m)t/T} = T\delta_{n,m}.$$

You can think of this as similar to $\widehat{e}_n \cdot \widehat{e}_m = \delta_{n,m}$; in that case we can write $\vec{f} = \sum_n f_n \widehat{e}_n$ and then dotting both sides with $\widehat{e}_m$ gives $f_m = \vec{f} \cdot \widehat{e}_m$. Writing out $e^{-2\pi i n/T}$ etc in terms of sin and cos, we can write the above simple formula in a more complicated looking, but fully equivalent way (remember, i is your pal):

$$\int_{t_0}^{t_0+T} dt \sin(2\pi n t/T)\sin(2\pi m t/T) = \int_{t_0}^{T} \cos(2\pi n t/T)\cos(2\pi m t/T) = \tfrac{1}{2}T\delta_{n,m},$$

$$\int_{t_0}^{t_0+T} dt \sin(2\pi n t/T)\cos(2\pi m t/T) = 0.$$

We then find

$$\tilde{f}_n = \int_{t_0}^{T+t_0} \frac{dt}{T} f(t) e^{2\pi i n t/T}.$$

Or equivalently

$$a_0 = \int_{t_0}^{T+t_0} \frac{dt}{T} f(t), \quad a_{m>0} = 2\int_{t_0}^{T+t_0} \frac{dt}{T} f(t)\cos(2\pi m t/T), \quad b_{m>0} = 2\int_{t_0}^{T+t_0} \frac{dt}{T} f(t)\sin(2\pi m t/T),$$

• Aside: can write it in QM notation (which is a convenient way to write basis vectors and their inner products): $|f\rangle = \sum_n \tilde{f}_n |n\rangle$, where $\tilde{f}_n = \langle n|f\rangle$ and $\langle n|m\rangle = \delta_{n,m}$. Now $\langle t|n\rangle = e^{-2\pi i n t/T}/\sqrt{T}$ are the normalized basis vectors in t-space and the notation encodes the fact that the inner product involves complex conjugation with $\langle n|t\rangle = \langle t|n\rangle^*$ generally complex. The orthogonality and completeness relations can be written as $\int_{t_0}^{T+t_0} dt|t\rangle\langle t| = \mathbf{1}$ and $\sum_n |n\rangle\langle n| = \mathbf{1}$, the unit operator (think of it as a matrix with 1's on the diagonals and 0's off-diagonal).

• Triangle wave example, using Mathematica to compute the Fourier coefficients. Approximate version of the function using M coefficients: plot for various M. Also play the sound.

• Square wave example.

• Gibbs phenomenon.

• Solving the forced, damped, SHO for general periodic functions via Fourier series.

• Fourier series for functions on a finite interval, via periodic extension. Examples with $f^{(p)}(t+T) = f(t)$, where $T = \Delta t$ of the original interval. Examples with $f^{(e)}(t) = f^{(e)}(t+2T)$, where the function in the range $t \in [t_0 - T, t_0]$ is defined via $f^{(e)}(t) = f(2t_0 - T)$, making it even upon reflection at t_0, so it has half as many jumps. Alternatively, take $f^{(o)}(t) = f^{(o)}(t+2T)$ with $f^{(o)}(t) = -f(2t_0 - t)$ in the range $[t_0 - T, t_0]$, so get sin's instead of cos's in expansion around t_0. Note the better falloff for the odd extension w.r.t. the mode number n, for fixed function $f(t)$, as compared with the original or the even extension.

• Solving boundary value problems using Fourier series, e.g. for the shaken spring demo.