11/6/19 Lecture outline

⋆ **Reading supplement for LSZ: Coleman lecture notes pages 140-175.**

• Last time:

$$G^{(n)}(x_1 \dots x_n) \equiv \langle \Omega | T\phi_H(x_1)\dots\phi_H(x_n)|\Omega\rangle = \frac{\langle 0|T\phi_{1I}(x_1)\dots\phi_{nI}(x_n)S|0\rangle}{\langle 0|S|0\rangle},$$

$$\widetilde{G}^{(n)}(p_1,\dots p_n) = \int \prod_{i=1}^{n} d^4x_i e^{-ip_ix_i} G^{(n)}(x_1\dots x_n).$$

$$\langle k_{n+1}\dots k_{n+m}|S-1|k_1k_2\dots k_n\rangle = \prod_{j=1}^{n+m}\frac{k_j^2-m_j^2}{i\sqrt{Z_j}}\widetilde{G}^{(n+m}(k_1,k_2,\dots,k_n,-k_{n+1},\dots -k_{n+m}),$$

where the $(n+m)$ multiplicative factors are to amputate the external propagators. The Z_j "wavefunction renormalization" factors will be discussed without details below, and examples will be discussed and computed next quarter when you learn about renormalization of diagrams with loops. Because the $(n+m)$ external states are taken on shell, they have $k_j^2 - m_j^2 \to 0$, so these multiplicative factors are $\sim 0^{n+m}$. These factors of zero cancel with factors of $1/0$ for diagrams with $n+m$ external propagators. Disconnected diagrams have fewer external propagators so they are set to zero by the 0^{n+m} multiplication. Consider for example $\widetilde{G}^{(4)}(k_1,k_2,k_3,k_4)$ for 4 external mesons in our meson-nucleon toy model. The lowest order contribution is at $\mathcal{O}(g^0)$ and is

$$(2\pi)^4\delta^{(4)}(k_1+k_4)\frac{i}{k_1^2-\mu^2+i\epsilon}(2\pi)^4\delta^{(4)}(k_2+k_3)\frac{i}{k_2^2-\mu^2+i\epsilon} + 2 \text{ permutations.}$$

This is the -1 that we subtract in $S-1$, and indeed would not contribute to $2\to 2$ scattering using the above formula, because it is set to zero by $\prod_{n=1}^4(k_n^2-m_n^2)$ when the external momenta are put on shell. To get a non-zero result, need a $\widetilde{G}^{(4)}$ contribution with 4 external propagators, which we get e.g. at $\mathcal{O}(g^4)$ with an internal nucleon loop in the toy model with cubic interactions, or at tree-level in $\lambda\phi^4$, where it gives $i\mathcal{A}_{\phi^4} = -i\lambda + \mathcal{O}(\lambda^2)$.

• Account for bare vs full interacting fields. Let $|k\rangle$ be the physical one-meson state of the full interacting theory, normalized to $\langle k'|k\rangle = (2\pi)^3 2\omega_k\delta^{(3)}(\vec{k}'-\vec{k})$. Then

$$\langle k|\phi(x)|\Omega\rangle = \langle k|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|\Omega\rangle = e^{ik\cdot x}\langle k|\phi(0)|\Omega\rangle \equiv e^{ik\cdot x}Z_\phi^{1/2}.$$

Can rescale the fields, s.t. $\langle k|\phi_R(x)|\Omega\rangle = e^{-ik\cdot x}$. The LSZ formula is:

$$\langle q_1\dots q_n|S-1|k_1\dots k_m\rangle = \prod_{a=1}^{n}\frac{q_a^2-m_a^2}{i\sqrt{Z}}\prod_{b=1}^{m}\frac{k_b^2-m_b^2}{i\sqrt{Z}}\widetilde{G}^{(n+m)}(-q_1,\dots -q_n,k_1,\dots k_m),$$

where the Green function is for the Heisenberg fields in the full interacting vacuum.

To derive the LSZ formula, consider wave packets, with some profile $F(\vec{k})$, and $f(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} F(\vec{k}) e^{-ik\cdot x}$, where we define $k_0 = \sqrt{\vec{k}^2 + \mu^2}$, so $f(x)$ solves the KG equation. Now define (with the understanding that $\phi \to \phi_R$ is the rescaled Heisenberg picture field, shifted if necessary to eliminate any classical vacuum expectation value, i.e. if $\Omega|\phi|\Omega = v$ then shift $\phi \to \phi - v$ in what follows) the time-dependent and spatially independent operator

$$\phi^f(t) = i\int d^3\vec{x}(\phi(\vec{x},t)\partial_0 f(\vec{x},t) - f(\vec{x},t)\partial_0\phi(\vec{x},t)).$$

Note that, since $f(x)$ satisfies the KG equation, can show

$$i\int d^4x f(x)(\partial^2 + \mu^2)\phi(x) = -\int dt \frac{\partial}{\partial t}\phi^f(t) = -\phi^f(t)|_{-\infty}^{\infty} \equiv \phi^f_{in} - \phi^f_{out}.$$

Show that $\phi^f(t)$ makes **single particle** wave packets from the vacuum, $\langle k|\phi^f(t)|\Omega\rangle = F(\vec{k})$ (note that the two terms add). Can similarly show (because of a relative minus sign), that $\langle\Omega|\phi^f(t)|k\rangle = 0$ (instead, $\phi^{f\dagger}$ annihilates the single particle state). Let $|n\rangle$ be an n-particle momentum eigenstate: $P^\mu|n\rangle = P^\mu_n|n\rangle$. Then $\langle n|\phi^f(t)|\Omega\rangle = \frac{\omega_{p_n}+p^0_n}{2\omega_{p_n}} F(\vec{p}_n) e^{-i(\omega_{p_n}-p^0_n)t}\langle n|\phi(0)|\Omega\rangle$, where $\omega_{p_n} \equiv \sqrt{\vec{p}_n^2 + \mu^2}$, which has $\omega_{p_n} < p^0_n$ for any multiparticle state. So for any state $\langle\psi|$ then $\lim_{t\to\pm\infty}\langle\psi|\phi^f(t)|\Omega\rangle = \langle\psi|f\rangle + 0$, where $|f\rangle \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k} F(k)\langle\psi|k\rangle$ is a one-particle state and the multiparticle states contributions sum to zero using the Riemann-Lebesgue lemma.

Make separated in states: $|f_n\rangle = \prod \phi^{f_n}(t_n)|\Omega\rangle$, and out states $\langle f_m| = \langle\Omega|\prod(\phi^{f_m})^\dagger(t_m)$, with $t_n \to -\infty$ and $t_m \to +\infty$. Then show

$$\langle f_m|S-1|f_n\rangle = \int \prod_n d^4x_n f_n(x_n) \prod_m d^4x_m f_m(x_m)^* \prod_r i(\partial_r^2 + m_r^2) G(x_n, x_m).$$

Take $f_i(x) \to e^{-ik_i x_i}$ at the end. There are 2^{n+m} terms with different $t_i \to \pm\infty$. Can show that almost all cancel and we are left only with the n in-states and the m out-states. See the Coleman lectures for more details in the case of $2 \to 2$.

• Next topic: quantization via the Feynman path integral. In canonical quantization, time plays a special role: equal time commutation relations, time ordered products, etc. Feynman found a completely new description of QM, by intuitively thinking about double slit interference and realizing that empty space can be thought of as being filled with screens that are full of holes, so such interference and taking multiple paths is always there. The path integral generalizes immediately from QM to QFT, and for different types of fields. Unlike canonical quantization, it makes Lorentz and Poincare symmetry manifest, and also gives a way to define QFT beyond perturbation theory. The classical limit is clarified, as the stationary phase limit of an integral. Similar statements apply in optics. The path integral also helps to connect QFT with statistical physics, with the path integral analogous to the partition function.

First consider QM:

$$U(x_b, T; x_a, 0) = \langle x_b | e^{-iHT/\hbar} | x_a \rangle = \int [dx(t)] e^{iS[x(t)]/\hbar}.$$

Integral can be broken into time slices, as way to define it.

E.g. free particle

$$\left(\frac{-im}{2\pi\hbar\epsilon}\right)^{N/2} \int \prod_{i=1}^{N-1} dx_i \exp[\frac{im}{2\hbar\epsilon} \sum_{i=1}^{N} (x_i - x_{i-1})^2]$$

Where we take $\epsilon \to 0$ and $N \to \infty$, with $N\epsilon = T$ held fixed.

Do integral in steps. Apply expression for real gaussian integral (valid: analytic continuation):

$$\int_{-\infty}^{\infty} d\phi \exp(ia\phi^2) = \sqrt{\frac{i\pi}{a}}.$$

where we analytically continued from the case of an ordinary gaussian integral. Think of a as being complex. Then the integral converges for $\mathrm{Im}(a) > 0$, since then it's damped. To justify the above, for real a, we need the integral to be slightly damped, not just purely oscillating. To get this, take $a \to a + i\epsilon$, with $\epsilon > 0$, and then take $\epsilon \to 0^+$. We'll see that this is related to the $i\epsilon$ that we saw in the Feynman propagator, which gave the T ordering.

After $n-1$ steps, get integral:

$$\left(\frac{2\pi i\hbar n\epsilon}{m}\right)^{-1/2} \exp[\frac{m}{2\pi i\hbar n\epsilon}(x_n - x_0)^2].$$

So the final answer is

$$U(x_b, x_a; T) = \left(\frac{2\pi i\hbar T}{m}\right)^{-1/2} \exp[im(x_b - x_a)^2/2\hbar T].$$

Note that the exponent is $e^{iS_{cl}/\hbar}$, where S_{cl} is the classical action for the classical path with these boundary conditions.

• Can derive the path integral from standard QM formulae, with operators, by introducing the time slices and a complete set of q and p eigenstates at each step.

$$\langle q', t'|q, t\rangle = \int\int \prod_{j=1}^{N} dq_j \langle q'|e^{-iH\delta t}|q_{N-1}\rangle\langle q_{N-1}|e^{-iH\delta t}|q_{N-2}\rangle \ldots \langle q_1|e^{-iH\delta t}|q\rangle,$$

where we'll take $N \to \infty$ and $\delta t \to 0$, holding $t' - t \equiv N\delta t$ fixed. Note that even though $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]+\ldots}$, we're not going to have to worry about this for $\delta t \to 0$: $e^{-iH\delta t} = e^{-i\delta t p^2/2m} e^{-i\delta t V(q)} e^{\mathcal{O}(\delta t^2)}$. Now note

$$\langle q_2|e^{-iH\delta t}|q_1\rangle = \int dp_1 \langle q_2|e^{-i\delta t p^2/2m}|p_1\rangle\langle p_1|e^{-iV(q)\delta t}|q_1\rangle,$$

$$= \int dp_1 e^{-iH(p_1,q_1)\delta t} e^{ip_1(q_2-q_1)}.$$

This leads to

$$\langle q', t'|q, t\rangle = \int [dq(t)][dp(t)] \exp(i\int_t^{t'} dt(p(t)\dot{q}(t) - H(p,q))),$$

and taking H quadratic in momentum and doing the p gaussian integral recovers the Feynman path integral.