12/3/19 Lecture outline

• Recap: we have discussed spin 0 and spin 1/2 quantum fields. Now move up to spin 1, i.e. the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group. Next quarter, you'll learn about renormalizability; there are associated complications with quantizing fields of spin greater than 1. The primary case is that of gauge fields: these are the force carriers. Aside: examples with spin 1 also include non-fundamental, composite fields, e.g. spin 1 mesons.

• Symmetries can be global or local. In the local case, they actually are not really symmetries but instead are associated with a redundant description of the physics: configurations differing by a local symmetry are physically identified. In the path integral description, we should not sum over configurations differing by a local symmetry: it would be over-counting. Global symmetries can be (and generally are) only approximate symmetries, whereas local symmetries must be exact (or the path integral would be ill defined). Examples of approximate global symmetries are charge conjugation, parity, and time reversal; these are also examples of discrete symmetries, and they are violated in the Standard Model by the weak interactions (though CPT is conserved, as follows from Lorentz symmetry). An example of a global symmetry is baryon number and lepton number conservation; they are preserved by the Standard Model Lagrangian. Some global symmetries are violated by quantum effects (anomalies, or more precisely instantons).

The local gauge symmetry of the Standard Model is $SU(3)_C \times SU(2)_W \times U(1)_Y$. The $SU(3)_C$ symmetry is confined. The $SU(2)_W \times U(1)_Y$ symmetry is spontaneously broken by a Bose condensate of the Higgs field down to $U(1)_{EM}$. Correspondingly, the $W^\pm$ and Z^0 force carriers are massive, and the photon is massless.

• The Dirac Lagrangian $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ preserves a global $U(1)$ symmetry $\psi \to e^{if}\psi$ with f a constant. The associated Noether conserved current is $j^\mu = \bar{\psi}\gamma^\mu\psi$, which of course transforms as a 4-vector under Lorentz transformations. For $m = 0$ the global symmetry is actually $U(1)_L \times U(1)_R$ acting on the $\psi_{L,R} = P_{L,R}\psi$, where $P_{L,R} = \frac{1}{2}(1 \pm \gamma_5)$.

Now consider the Dirac Lagrangian for N Dirac Fermions $\mathcal{L} = \sum_{a=1}^{N} \bar{\psi}^a(i\partial\!\!\!/ - m_a)\psi_a$. More generally, the mass m could be a $N \times N$ matrix; here we are taking it to be diagonal. If the m_a are all different, there is a $U(1)^N$ symmetry. If the m_a are all the same, it enhances to $U(N)$: $\psi_a \to U_a^b\psi_b$ with $U^\dagger U = 1$. For $N > 1$ this symmetry is non-Abelian $U_1U_2 \neq U_2U_1$.

• Consider the case of $N = 1$ massive Dirac Fermion. To make $U(1)$ into a local symmetry, we want to allow general $f(x)$. Then we need to replace $\partial_\mu \to D_\mu \equiv \partial_\mu + ieqA_\mu$, where eq is the charge of the field ψ (we can redefine $eq \to q$ but I am writing it this way because e.g. e is the charge of the electron and $\alpha = e^2/4\pi\hbar c \sim 1/137$ plays the role of a coupling constant in quantum electrodynamics, and q is observed to be quantized to be integers). Under the local $U(1)$ transformation $\psi \to e^{-ieqf(x)}\psi$ and $eA_\mu \to eA_\mu + \partial_\mu f$, and then $D_\mu\psi \to (\partial_\mu + ieq(A_\mu + \partial_\mu f)(e^{-ieqf}\psi) = e^{-ieqf}D_\mu\psi$ transforms covariantly (i.e. nicely, with just an overall phase).

So the Dirac equation with a local gauge symmetry is $\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi = \bar{\psi}(i\not{\partial} - m)\psi - A_\mu j^\mu$ where $j^\mu = eq\bar{\psi}\gamma^\mu\psi$ is the associated current. This is the usual way that electromagnetic currents couple in a Lagrangian density. The Lagranian is invariant under a gauge transformation $\delta A_\mu = \partial_\mu f$ thanks to current conservation $\partial_\mu j^\mu = 0$.

Next step: include kinetic terms for A_μ (corresponding to the Maxwell Lagrangian) and then quantize A_μ.

• Consider a Lagrangian for a spin 1 quantum field A_μ. The components of A_μ will satisfy something like a KG equation, being massive or massless. We'll start with the massive case first, as a warmup for the massless case. Physically, this could be e.g. the Z^μ massive vector bosons of the electroweak force.

For the massive vector mesons, write down the general lagrangian:

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu A^\nu \partial_\mu A^\nu + a\partial_\mu A^\mu \partial_\nu A^\nu + bA_\mu A^\mu).$$

The sign is chosen so that the kinetic terms of the spatial components have the right sign. Write the EOM:

$$-\partial^2 A_\nu - a\partial_\nu(\partial \cdot A) + bA_\nu = 0,$$

and note plane wave solutions $A_\mu(x) = \epsilon_\nu e^{-ik\cdot x}$ solves it if $k^2\epsilon_\nu + ak_\nu(k\cdot\epsilon) + b\epsilon_\nu = 0$. The longitudinal solutions have $\epsilon^\mu \propto k^\mu$ and have mass $\mu_L^2 = -b/(1+a)$. The transverse have mass $\mu_T^2 = -b$. Can eliminate the uninteresting longitudinal solution by taking $a = -1$ and $b \neq 0$, then write Proca lagrangian in terms of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\mu^2 A_\mu A^\mu$$

Each component A_μ satisfies the KG equation with mass μ. Can choose $\epsilon^{(\pm)} = \frac{1}{\sqrt{2}}(0, 1, \mp i, 0)$ and $\epsilon^{(0)} = (0, 0, 0, 1)$, where the label is the value of J_z of the spin 1 vector. Normalize by $\epsilon^{(r)*} \cdot \epsilon^{(s)} = -\delta^{rs}$ and $\sum_r \epsilon_\mu^{(r)*}\epsilon_\nu^{(r)} = -g_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2}$.

The conjugate momenta to A_μ are $\pi^0 = \partial\mathcal{L}/\partial\dot{A}_0 = 0$, and $\pi^i = \partial\mathcal{L}/\partial\dot{A}_i = -F^{0i} = E^i$. Then $\mathcal{H} = -\frac{1}{2}(F_{0i}F^{0i} - \frac{1}{2}F_{ij}F^{ij} + \mu^2 A_i A^i - \frac{1}{2}\mu^2 A_0 A^0) \geq 0$.