12/5/19 Lecture outline

• Last time: made the $U(1)$ global symmetry of the Dirac equation into a local symmetry by replacing $\partial_\mu \to D_\mu \equiv \partial_\mu + iqA_\mu$, where q is the charge of the field ψ: $\mathcal{L} \supset \bar{\psi}(i\not{D} - m)\psi$ with local gauge "symmetry" (really redundancy) $\psi \to e^{-iqf(x)}\psi$ and $A_\mu \to A_\mu + \partial_\mu f$, since $D_\mu \psi \to e^{-iqf} D_\mu \psi$ transforms covariantly.

Noted that $\mathcal{L} \supset -A_\mu j^\mu$ and this is gauge invariant as long as j^μ is conserved.

We then wrote down kinetic terms for a massive spin 1 field A_μ and noted that they decompose into separate longitudinal and transverse terms, with different masses, and we can arrange to decouple the longitudinal term.

• Continue from there and, to streamline the discussion, consider several cases in parallel:

$$(0) \qquad \mathcal{L} \supset -\tfrac{1}{2}(\partial_\mu A^\nu \partial_\mu A^\nu + a\partial_\mu A^\mu \partial_\nu A^\nu + bA_\mu A^\mu).$$

$$(i) \qquad \mathcal{L} \supset \tfrac{1}{2}(-\partial_\mu A^\nu \partial^\mu A_\nu + \partial_\mu A^\mu \partial_\nu A^\nu + m_A^2 A_\nu A^\nu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}m_A^2 A_\mu A^\mu.$$

$$(ii) \qquad \mathcal{L} \supset -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad A_\mu \sim A_\mu + \partial_\mu f$$

$$(iii) \qquad \mathcal{L} \supset -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial \cdot A)^2 \qquad (A_\mu \sim A_\mu + \partial_\mu f)$$

Case (0) was discussed last time. The longitudinal part corresponds to $A_\mu \sim \partial_\mu \phi$ and doesn't add anything beyond essentially just including a scalar field ϕ – so we won't discuss it further.

Case (i) decouples the longitudinal mode by making its mass infinite, and is the Lagrangian for a massive vector field. We do not identify under gauge equivalence, and indeed the mass term is not gauge invariant. We will see that it has 3 physical, propagating degrees of freedom.

Simply taking the mass $m_A \to 0$ in case (i) gives a sick theory. If we want $m_A \to 0$, we must impose gauge equivalence as in case (ii) – which is what we wanted to consider anyway for gauge symmetry. This theory has two physical propagating degrees of freedom, as in the two polarizations of light.

Case (iii) includes what is called a gauge fixing term, with gauge parameter ξ. It is essentially case (0) with $b \to 0$. Before imposing gauge equivalence (which is in parentheses because it is only imposed at the end) the theory has 4 polarizations and is a sick: some of the polarizations have the wrong sign kinetic term. Nevertheless, it is sometimes a useful way to treat case (ii), by temporarily keeping the unwanted d.o.f. and then verifying that they decouple and throwing them away at the end of the calculation. The decoupling amounts to verifying that physical quantities – sums of Feynman diagrams but not necessarily individual diagrams – are independent of ξ.

• Let's start with case (i). As in E and M we can write it in terms of $F^{i0} = E^i$ with $\vec{E} = -\nabla A^0 - \dot{\vec{A}}$ and $F^{ij} = \epsilon^{ijk} B_k$ with $\vec{B} = \nabla \times \vec{A}$ as $\mathcal{L} \supset \frac{1}{2}(\vec{E}^2 - \vec{B}^2) + \frac{1}{2} m_A^2 (A_0^2 - \vec{A}^2)$. Note that A_0 is non-dynamical, and its equation of motion is Gauss' law. The conjugate momenta to A_μ are $\pi^0 = \partial \mathcal{L} / \partial \dot{A}_0 = 0$, and $\pi^i = \partial \mathcal{L} / \partial \dot{A}_i = -F^{0i} = E^i$. Then $\mathcal{H} = -\frac{1}{2}(F_{0i} F^{0i} - \frac{1}{2} F_{ij} F^{ij} + \mu^2 A_i A^i - \frac{1}{2} m_A^2 A_0 A^0)$. The plane wave solutions are $A^\mu \sim \epsilon^\mu e^{-ikx} + h.c.$. In the frame where $k^\mu = (k^0, \vec{0})$, we can choose $\epsilon^{(\pm)} = \frac{1}{\sqrt{2}}(0, 1, \mp i, 0)$ and $\epsilon^{(0)} = (0, 0, 0, 1)$, where the label is the value of J_z of the spin 1 vector. Normalize by $\epsilon^{(r)*} \cdot \epsilon^{(s)} = -\delta^{rs}$ and $\sum_r \epsilon_\mu^{(r)*} \epsilon_\nu^{(r)} = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m_A^2}$.

• Quantize the massive vector:

$$[A_i(t, \vec{x}), F^{j0}(t, \vec{y})] = i \delta_i^j \delta^{(3)}(\vec{x} - \vec{y}).$$

Note that we only quantize the space components; we do not directly quantize the non-dynamical $A_0(t, \vec{x})$. Instead, we solve for A_0 by the EOM, which here gives $\nabla \cdot (-\nabla A_0 - \dot{\vec{A}}) = m_A^2 A_0$, and hence $A_0(t, \vec{x}) = \int d^3 \vec{x}' e^{-m_A |\vec{x} - \vec{x}'|} \frac{\nabla \cdot \dot{\vec{A}}(t, \vec{x})}{4\pi |\vec{x} - \vec{x}'|}$. So A_0 is a complicated, composite operator, with some commutation relation determined by that of A_i; we will not bother to write it down. (In the massless case, we can choose a gauge where $A_0 = \nabla \cdot \vec{A} = 0$ and the physics is gauge invariant.) In terms of the plane wave solutions,

$$A_\mu(x) = \sum_{r=1}^{3} \int \frac{d^3 k}{(2\pi)^3 (2\omega_k)} \left[a_k^r \epsilon_\mu^r e^{-ikx} + a_k^{\dagger r} \epsilon_\mu^{*r} e^{ikx} \right],$$

(as usual, there is a choice of convention in the normalization of the creation and annihilation operators), and with this normalization the quantization condition implies that

$$[a_k^r, a_{k'}^{\dagger s}] = \delta^{rs} (2\pi)^3 (2\omega_k) \delta^3(\vec{k} - \vec{k}').$$

and

$$: \mathcal{H} := \sum_r \int \frac{d^3k}{(2\pi)^3(2\omega_k)} \omega_k a_k^{\dagger r} a_k^r.$$

• The propagator, the contraction of $A_\mu(x)$ and $A_\nu(y)$, is

$$\langle T A_\mu(x) A_\nu(y) \rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \left[\frac{-i(g_{\mu\nu} - k_\mu k_\nu/m_A^2)}{k^2 - m_A^2 + i\epsilon} \right].$$

This is obtained from the above expansion of A_μ and the commutation relations for the creation and annihilation operators.

In the path integral description, we obtain the propagator from the generating functional with sources $s_\mu(x)$ for A^μ, so we get correlation functions for A_μ via acting with $A_\mu \leftrightarrow -i\frac{\delta}{\delta s^\mu}$ on

$$Z[s_\mu] = \int [dA_\mu] \exp(i \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2} m_A^2 A_\mu A^\nu - s_\mu A^\mu \right).$$

Note that s^μ couples like a conserved current J^μ. Note also that we can integrate by parts to write $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_A^2 A_\mu A^\mu = \frac{1}{2}A^\mu(\partial_\mu\partial_\nu - g_{\mu\nu}(\partial^2 + m_A^2))A^\nu$ and we can do the quadratic functional integral over A^μ by completing the square, as usual, to get

$$Z[s_\mu] = \exp(-\tfrac{1}{2} \int d^4x \int d^4y s_\mu(x) D^{\mu\nu}(x-y) s_\nu(y)),$$

where $(\partial_\mu\partial_\nu - g_{\mu\nu}(\partial^2 + m_A^2))D^{\mu\kappa} = i\delta_\mu^\kappa \delta^4(x-y)$ is the Green's function for the differential operator in the original integral, and $D^{\mu\nu}(x-y)$ is equal to $\langle T A_\mu(x) A_\nu(y) \rangle$.

So the Feynman rule is that massive vectors have the momentum space propagator

$$\left[\frac{-i(g_{\mu\nu} - k_\mu k_\nu/m_A^2)}{k^2 - m_A^2 + i\epsilon} \right].$$

And $\langle 0 | A_\mu(x) | V(k,r) \rangle = \epsilon_\mu(k)^r e^{-ikx}$, so incoming vector mesons have $\epsilon_\mu^r(k)$ and outgoing have $\epsilon^{*r}(k)$.

We can couple the massive vector to other fields, e.g. to a fermion via $\mathcal{L}_{int} = -g\bar{\psi}\not{A}\Gamma\psi$, with $\Gamma = 1$ (vector) or $\Gamma = \gamma_5$ (axial vector). Gives Feynman rule that a vertex has a factor of $-ig\gamma^\mu\Gamma$.

• Now consider the massless theory, $m_A \to 0$. If we add $\mathcal{L} \supset -A_\mu j^\mu$ to the massive theory, get $\partial_\mu A^\mu = m_A^{-2}\partial_\mu j^\mu$, so there is only a sensible limit if $\partial_\mu j^\mu = 0$, must couple to a conserved current. The only way to have a sensible $m_A \to 0$ limit is if A_μ is a gauge field, associated with a local gauge symmetry. Indeed, the operator in brackets in

$$[\eta_{\mu\nu}(\partial^\rho \partial_\rho) - \partial_\mu \partial_\nu]A^\nu = 0$$

is not invertable: it annihilates any function of form $\partial^\mu \lambda$. Solution: require that $A_\mu \sim A_\mu + \partial_\mu \lambda$, i.e. gauge invariance. The space of gauge fields has equivalent gauge orbits.

Minimal coupling examples:

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi = \bar{\psi}(i\not{\partial} - q\not{A} - m)\psi.$$

$$\mathcal{L} = D_\mu \phi^* D^\mu \phi - m^2 |\phi|^2.$$

The first gives a $\bar{\psi} A_\mu \psi$ Feynman vertex weighted by $-iq\gamma^\mu$, and the second gives a $\phi^*(p')A_\mu \phi(p)$ vertex weighted by $ieq(p+p')^\mu$, along with a $A_\mu A_\nu \phi^* \phi$ seagull graph weighted by $2iq^2 g^{\mu\nu}$ (factor of 2 because of the two identical A_μ fields).

As in the massive vector case, A_0 has no kinetic term, can solve its EOM ($\nabla \cdot \vec{E} = 0 \to \nabla^2 A_0 + \nabla \cdot \dot{\vec{A}} = 0$):

$$A_0(\vec{x}) = \int d^3\vec{x}' \frac{\nabla \cdot \dot{\vec{A}}(\vec{x}')}{4\pi |\vec{x} - \vec{x}'|}.$$

Gauge fixing: can always choose e.g. $\partial_\mu A^\mu = 0$. Doesn't entirely fix the gauge. Can still pick $\nabla \cdot \vec{A} = 0$ – Coulomb gauge – then $A_0 = 0$. See two polarizations. So take $\vec{\epsilon}^r$ with $\vec{\epsilon}_r \cdot \vec{p} = 0$, orthonormal. The completeness relation is similar to that above, except that we replace $\mu^2 \to |\vec{p}|^2$. The propagator is then

$$\langle T A_i(x) A_j(y) \rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \left[\frac{i(\delta_{ij} - k_i k_j / |\vec{k}|^2)}{k^2 + i\epsilon} \right].$$

This gauge can be a pain in the interacting theory (need to write instantaneous $\delta(x^0 - y^0)/|\vec{x} - \vec{y}|$ Coulomb interaction). It's nicer to write something more manifestly Lorentz invariant.

In the massive vector case, we had the propagator $-i(g_{\mu\nu} - k_\mu k_\nu / m_A^2)/(k^2 - m_A^2 + i\epsilon)$. In the $m_A \to 0$ massless gauge theory, gauge invariance ensures that the $k_\mu k_\nu$ term has no effect in physical, on-shell amplitudes.

- Gauge fixing. Try to preserve Lorentz invariance by imposing $\partial_\mu A^\mu = 0$, and not $A_0 = 0$. Can modify $\mathcal{L}$ to get Lorentz gauge EOM. More generally, can consider

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial \cdot A)^2,$$

and quantize for any parameter ξ. Popular choices are $\xi = 1$ (Feynman) and $\xi = 0$ (Landau). Now get $\pi^0 = \partial\mathcal{L}/\partial(\dot{A}_0) = -\partial_\mu A^\mu/\xi$. Do canonical quantization for all components, $[A_\mu(\vec{x}), \pi_\nu(\vec{y})] = i\eta_{\mu\nu}\delta(\vec{x}-\vec{y})$. Write plane wave expansion with 4 polarizations, normalized to $\epsilon^\lambda \cdot \epsilon^{\lambda'} = \eta^{\lambda\lambda'}$. Get that timelike polarizations create negative norm states. Can fix this by imposing $\partial^\mu A_\mu^+|\Psi\rangle = 0$ on the physical states, along with gauge invariance relation, to get a physical Hilbert space with positive norms.

Propagator for gauge field is

$$\langle T A_\mu(x) A_\nu(y)\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \left[\frac{-i(g_{\mu\nu} + (\xi - 1)k_\mu k_\nu/k^2)}{k^2 + i\epsilon}\right].$$

Again, the $k_\mu k_\nu$ piece will drop out in the end in physical amplitudes. Just need to make a choice and stick with it consistently. Or keep ξ as a parameter, and then it's a good check on the calculation that the ξ indeed drops out in the end.

• QED, and examples. Recall the Feynman rules: incoming electrons get a $u^r(p)$, outgoing electrons get a $\bar{u}^r(p)$, incoming positrons get a $\bar{v}^r(p)$, outgoing positrons get a $v^r(p)$. Incoming photons get an $\epsilon^\mu(p)$, and outgoing photons get a $\epsilon^{*\mu}(p)$. The electron propagator is $i/(\not{p} - m + i\epsilon)$, and the photon propagator is $-i(g_{\mu\nu} + (\xi - 1)(k_\mu k_\nu/k^2))/(k^2 + i\epsilon)$. The interaction vertex is $-ie\gamma^\mu$.

Consider for example Compton scattering, $e^-(p)\gamma(k) \to e^-(p')\gamma(k')$ (related to $e^+e^- \to \gamma\gamma$ by crossing symmetry):

$$i\mathcal{A} = -ie^2\epsilon^*_\mu(k')\epsilon_\nu(k)\bar{u}^{r'}(p')\left[\frac{\gamma^\mu(\not{p} + \not{k} + m)\gamma^\nu}{(p+k)^2 - m^2} + (k \to -k').\right]u^r(p)$$

$$= -i\epsilon^*_\mu(k')\epsilon_\nu(k)\bar{u}^{r'}(p')\left[\frac{\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + (k \to -k')\right]u^r(p).$$

Writing $i\mathcal{A} = \mathcal{M}^{\mu\nu}\epsilon_\nu(k)\epsilon^*_\mu(k')$, you can verify that $k_\mu\mathcal{M}^{\mu\nu} = 0$. The amplitude must always vanish if $\epsilon^\mu \sim k^\mu$, and this can be understood as a consequence of current conservation, thinking about the current as the source for A^μ.

To compute the differential cross section, we square this and multiply it by the $2 \to 2$ phase space factor. It simplifies to sum over final state spins and average over initial state ones, since then we can use the completeness relations for the external spinors or polarizations.

As another example, consider $e^-(p)e^+(q) \to e^-(p')e^+(q')$ at tree level. There is then both an s-channel and a t-channel diagram. Or we could consider $e^+e^- \to \mu^+\mu^-$ and then there is only the s-channel diagram. The s-channel term is

$$i\mathcal{A} = \bar{u}^{r'}(p')(-ie\gamma^\mu)v^{s'}(q')\frac{-i}{k^2}(g_{\mu\nu} - \frac{(1-\xi)k_\mu k_\nu}{k^2})\bar{v}^s(q)(-ie\gamma^\nu)u^r(p)$$

with $k = p + q = p' + q'$. We can now verify that this is independent of ξ, since e.g. $\bar{v}^2(q)\not{k}u^r(p) = 0$, thanks to $\not{p}u^r(p) = mu^r(p)$ and $\bar{v}^s(q)\not{q} = -m\bar{v}^s(q)$.

Other examples, $e^+e^- \to e^+e^-$ vs $e^-e^- \to e^-e^-$. The two are related by crossing symmetry. Mention $e^-e^\mp \to e^-e^\mp$ and the Coulomb potential: opposites attract and same sign charges repel. Contrast this with the scalar Yukawa case, where the potential is always attractive. Because here $\bar{v}\gamma^0 v \to +2m$, whereas in the scalar case got $\bar{v}v \to= -2m$.