10/28/19 Lecture outline

• Last time: Warmup: consider quantum mechanics, with $U(t) = Te^{-i\int^t H(t)dt}$,

$$\langle f|U(t)|i\rangle \approx -i\langle f|H_{int}|i\rangle \int_0^t dte^{i\omega t},$$

where $\omega = E_f - E_i$. If we take $t \to \infty$ first, we get $\delta(\omega)$ and squaring would give nonsense. That's because we're asking the wrong question if we ask about probability for a transition over all time – instead, we should ask about the rate. So keep t finite for now. Squaring gives $P(t) = 2|\langle f|H_{int}|i\rangle|^2(1-\cos\omega t)/\omega^2$. For $t \to \infty$, multiply by $dE_f\rho(E_f)$ and replace $(1-\cos\omega t)/\omega^2 = 4\sin^2(\frac{1}{2}\omega t)/\omega^2 \to \pi t\delta(\omega)$ (using $\int_{-\infty}^{\infty} dx x^{-2}\sin^2 x = \pi$ (hint: $\sin^2 x/x^2 = (2 - e^{i2x} - e^{-i2x})/4x^2$ and close the contour in the correct directions)) to get

$$\dot{P}_{i\to f} = 2\pi|\langle f|H_{int}|i\rangle|^2\rho(E).$$

This is "Fermi's Golden Rule" – it was actually derived by Dirac, but Fermi used it a lot and called it the golden rule. Another aside: Fermi and Dirac independently discovered that spin 1/2 objects must anticommute, and Dirac generously named such objects "Fermions".

Naively taking $t \to \infty$ initially would have given amplitude $\sim \delta(\omega)$, and squaring that would give $\delta(\omega)^2$, which needs to be replaced with $\delta(\omega)2\pi T$, and then divide by T to get the rate. Similarly in field theory, $\delta(p)^2$ should be replaced with probability $\sim \delta(p)$ times phase space volume factors.

• Compute probabilities by squaring the S-maxtrix amplitudes, but have to be careful with the delta functions, since squaring the delta functions would give nonsense. In QM we found $\dot{P}_{i\to f} = 2\pi|\langle f|H_{int}|i\rangle|^2\rho(E)$ "Fermi's golden rule."

• Phase space factors. Put the system in a box of spatial volume V, and time interval T; as a check, the factors of V and T should cancel in physical results. There are $Vd^3\vec{k}/(2\pi)^3$ momentum states with $\vec{k}$ in the range $d^3\vec{k}$ and $TdE/2\pi$ energy states in the range dE. So delta-functions squared in 3-momentum and 4-momenta are replaced as

$$(2\pi)^3\delta^3(\vec{p}=0) \to V, \qquad (2\pi)^4\delta^4(p^\mu = 0) \to VT$$

since e.g. $\int d^3\vec{x}e^{i\vec{p}\cdot\vec{x}} = (2\pi)^3\delta^3(\vec{p})$ gives for $\vec{p}=0$: $\int d^3x 1 = V$. The differential probability per unit time of a process is

$$\frac{dP}{T} = \frac{|\langle f|(S-1)|i\rangle|^2}{T\langle i|i\rangle\langle f|f\rangle}\prod_f \frac{V}{(2\pi)^3}d^3p_f = \frac{1}{\langle i|i\rangle\langle f|f\rangle}|\mathcal{A}_{fi}|^2(2\pi)^4V\delta^4(p_f - p_i)\prod_f \frac{V}{(2\pi)^3}d^3p_f.$$

Particle states are normalized as $\langle k|k\rangle = (2\pi)^3 2E_k\delta^3(0) \to 2E_kV$, so

$$\frac{dP}{T} = \frac{V}{\prod_i(2E_iV)}|\mathcal{A}_{fi}|^2 d\Pi_{LIPS}, \qquad d\Pi_{LIPS} \equiv (2\pi)^4\delta^4(p_f - p_i)\prod_f \frac{1}{(2\pi)^3(2E_f)}d^3p_f$$

where $d\Pi_{LIPS}$ is the Lorentz invariant phase space for the final states, and it is independent of V. Verify units: $[\dot{P}] = 2(4 - n_{i,tot} - n_{f,tot}) - 3 + 2n_{i,tot} - 4 + 2n_{f,tot} = 1$, good.

There are two cases to consider for the initial state: a one-body initial states for a lifetime decay rate, or a two-body initial state for a scattering cross section. Decays: differential decay probability per unit time: $d\Gamma = \frac{1}{2M}|\mathcal{A}_{fi}|^2 d\Pi_{LIPS}$. As expected, it is independent of V and T. Integrate over all possible final states to get $\Gamma = 1/\tau$ where τ is the lifetime.

The differential scattering cross section is the scattering number probability per unit time and flux:

$$d\sigma = \frac{dP}{T\Phi} = \frac{|\mathcal{A}_{fi}|^2}{4E_1E_2V\Phi}d\Pi_{LIPS}, \qquad V\Phi = |\vec{v}_{rel}| = |\vec{v}_1 - \vec{v}_2|,$$

where the density is taken to be one particle per volume V and, as required, V indeed cancels in the final result. Note that the result is relativistically invariant. Write $dNdt = (d\sigma|\vec{v}_1 - \vec{v}_2|\rho_1\rho_2)(Vdt)$, the LHS is the number of collisions, which should be the same in any frame, and the factor (Vdt) on the RHS is relativistically invariant. For simplicity we take $\vec{v}_1$ and $\vec{v}_2$ to be parallel, $\vec{v}_1 \times \vec{v}_2 = 0$. We want $d\sigma$ to be defined to be the cross section in the rest frame of one of the particles, so we want to define it to be boost invariant. So we need to show that $|\vec{v}_1 - \vec{v}_2|\rho_1\rho_2$ is boost invariant; in the rest frame of particle 2 it reduces to $v_{rel}\rho_1\rho_2$, which is what we want. Let's just check it. Under a boost to a frame with relative velocity u (taken along the direction of $\vec{v}_1$ and $\vec{v}_2$, we have $v_i \to (v_i + u)/(1 + v_iu)$ and $\rho_i \to \rho_i\gamma_u(1 + v_iu)$ (recall $J_i^\mu = \rho_i(1, \vec{v}_i)$ transforms as a 4-vector). Find that $|\vec{v}_1 - \vec{v}_2|\rho_1\rho_2$ is indeed invariant under boosts in the direction along $\vec{v}_1 - \vec{v}_2$. Indeed, if we take p_1 and p_2 to both point along the 1 axis, then $E_1E_2|\vec{v}_1 - \vec{v}_2| = |\epsilon^{23\mu\nu}p_{1,\mu}p_{2,\nu}|$, which shows that it transforms like an area element $dx^2 \wedge dx^3$. For our application, we define $\rho_i = 1/V$ in the lab frame.

Two body final states (in CM frame): $D = \int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}(2\pi)^4\delta^3(\vec{p}_1 + \vec{p}_2)\delta(E_1 + E_2 - E_T)$ gives

$$D = \int \frac{1}{(2\pi)^3 4E_1E_2}p_1^2 dp_1 d\Omega_1 (2\pi)\delta(E_1 + E_2 - E_T).$$

Using $E_1 = \sqrt{p_1^2 + m_1^2}$ and $E_2 = \sqrt{p_1^2 + m_2^2}$ get $\partial(E_1 + E_2)/\partial p_1 = p_1E_T/E_1E_2$ and finally $D = p_1 d\Omega_1/16\pi^2 E_T$. This should be divided by 2! (more generally, n!) if the final states are identical.