Physics 220, Spring 2004, Homework 3, fun with O, Due May 6

1. Write the 24 elements of O as S_4 permutations of the 4 body diagonals (between opposite edges). Label the vertices of the bottom face of the cube as 1,2,3,4, with 1 and 3 diagonally across from each other and 2 and 4 diagonally across from each other. Body diagonal 1 connects vertices 1 and 8, body diagonal 2 connects vertices 2 and 7, body diagonal 3 connects vertices 3 and 6, and body diagonal 4 connects vertices 4 and 5 (vertices 5,6,7,8 are on the top face of the cube). Write each O group elements as particular permutations of the eight vertices (as a subgroup of S_8) and show that each corresponds to a particular distinct S_4 permutation element of the 4 body diagonals.

2. The representation F_1 of O is how a vector, such as $\vec{r}$, transforms under this discrete subgroup of the rotation group[1]. In this problem you will do some checks that (yz, xz, xy) gives the other 3 dimensional representation, F_2.
   Consider the examples of C_2 and C_4 elements that acts on F_1 as:

$$C_2: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad C_4: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

If we put center the cube at the origin, with edges parallel to the $\hat{x}$, $\hat{y}$, and $\hat{z}$ axes, the above C_2 is a π rotation around the line $x = y$, and C_4 is a rotation by $2\pi/4$ around the $\hat{z}$ axis. You can easily verify that the traces of the above matrices, and also C_4^2 (squaring the above C_4 matrix), agree with the characters in the F_1 irrep of the C_2, C_4, and C_4^2 conjugacy classes, as given in lecture.

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[1] The cross product of two vectors $\vec{v} \times \vec{w}$ also transforms in the representation F_1, because it transforms exactly the same as an ordinary vector under all proper rotations. Cross products do transform unusually under improper rotations, such as reflection in a plane or inversions. E.g. an inversion takes $\vec{v} \to -\vec{v}$ and $\vec{w} \to -\vec{w}$, but $\vec{v} \times \vec{w} \to +\vec{v} \times \vec{w}$. Ordinary vectors are called polar, and the cross product of two polar vector gives what's called an axial vector, which transforms the same under all proper rotations, but oppositely under reflections in a plane or inversions. Examples of polar vector are position, force, and electric field. Examples of polar vectors are angular momentum, torque, and magnetic field. Since O contains only proper rotations, both polar and axial vectors transform as the same irrep, namely F_1. On the other hand, O is isomorphic to T_d, the tetrahedral group augmented by a reflection in a plane; in this description the two 3d irreps *do* correspond to polar and axial vectors.

Write out the 3×3 matrices for C_2, C_4, and C_4^2 acting on the basis (yz, xz, xy) and verify that the characters of these are as given in lecture for the F_2 irrep.

3. A system, has Hamiltonian $H = H_0 + H_1$, with H_0 having symmetry group O, and perturbation H_1 preserves the subgroup T. As in lecture, call the T irreps: A_0 (trivial), E (sum of two complex conjugate 1d irreps), and F (tcorresp. to $\vec{r}$). Call the O irreps: A_0 (trivial), A_1 (another 1d irrep), E (2d irrep), F_1 (corresp. to $\vec{r} = (x, y, z)$) and F_2 (corresp. to (yz, xz, xy)).

   a) Find how each O irrep splits into T irreps.

   b) Suppose that the first five energy levels of H_0 correspond to irreps A_0, A_1, E, F_1, and F_2, in that order. The small perturbation H_1 changes the energy of a O level, with the T irreps affected as: lowered for A_0, raised for F, and no change for E. Draw a qualitative picture of the first five H_0 energy levels, and their splitting due to H_1, indicating the degeneracies of the levels before and after the splitting. (Draw the splitting size as small compared to the initial separation of the first five H_0 energy levels.)

4. Consider again a system with H_0 having the symmetry group O. Find the selection rules for when a photon is emitted (dipole radiation, with H_1 in the rep. corresponding to $\vec{r}$). List all possible leading order transitions, e.g. can one have a $F_2 \leftrightarrow A_0$ transition? Etc. For each of the five possible initial irreps, list which of the 5 final possible irreps it can transition to upon emitting a photon.

5. Consider again a system with H_0 having symmetry group O and now find the selection rules for leading order radiation due to quadrupole radiation (a pair of photons in the $l = 2$ state). Recall Q_{ij} transforms as the symmetrized product of two vectors $\vec{v}$ and $\vec{w}$, i.e. $v_i w_j + v_j w_i$) omitting the trace part (the dot product $\vec{v} \cdot \vec{w}$). So consider $F_1 \otimes F_1$ and subtract out the A_0 irrep (this is $\vec{v} \cdot \vec{w}$), and also subtract out antisymmetric part, corresponding to the cross product $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$ (as discussed in the footnote, this is the F_1 irrep), and what's left is the O representation of the 5 quadrupole states Q_{ij}, as a sum of O irreps. Now use this information to find the selection rules for all of the allowed transitions due to quadrupole radiation.