4/17/07 Lecture 5 outline

• The time evolution of a free particle is given by

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}} e^{i(px-Et)/\hbar}\phi(p), \qquad \phi(p) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}\psi(x,t=0).$$

We can also write $\phi(p,t) = \phi(p)e^{-iE(p)t/\hbar}$, so $\psi(x,t)$ is the Fourier transform of $\phi(p,t)$.

• Suppose $\phi(p)$ is peaked at $p = p_0$, with width Δp. Then expand $E(p) = E(p_0) + (p - p_0)\frac{dE}{dp} + \frac{1}{2}(p-p_0)^2\frac{d^2E}{dp^2} + \ldots$. Let $\beta \equiv \frac{1}{2}d^2E/dp^2$, then for $t \ll 1/\beta(\Delta p)^2$, find $|\psi(x,t)|^2 \approx |\psi(x - v_g t, 0)|^2$: packet moves with velocity given by the group velocity, $v_g = d\omega/dk$, which for us is now $v_g = dE/dp = p/m$, which is indeed reasonable: the group velocity equals the particle's velocity. The spreading of wave packets is generally given by $d^2\omega/dk^2$, which for us is d^2E/dp^2, which is $1/m$ for a non-relativistic particle.

• For example, suppose

$$\psi(x,t=0) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{1/2} e^{ip_0x/\hbar}\exp(-x^2/4\sigma^2) \to |\phi(p)|^2 = \frac{1}{\sqrt{2\pi}\tilde{\sigma}}\exp(-(p-p_0)^2/2\tilde{\sigma}^2),$$

which is of the Gaussian form, with average momentum p_0. Yields

$$\psi(x,t) = \left(\frac{1}{\sqrt{2\pi}(\sigma + it\hbar/2m\sigma)}\right)^{1/2} e^{ip_0(x-p_0t/2m)/\hbar}\exp(-(x-p_0t/m)^2/4(\sigma+it\hbar/2m\sigma)^2).$$

Another gaussian. In particular, $|\psi(x,t)|^2$ is a gaussian centered at $\langle x\rangle = p_0t/m$, with width $\sigma(t) = \sqrt{\sigma^2 + (\hbar t/2m\sigma)^2}$. Spreading of the packet (just like diffraction): the more the particle is localized, the more it will later spread out. E.g. an electron, localized initially in $\sigma = 10^{-10}m$ spreads to 1AU ($= 8 \times 60 \times 3 \times 10^8 m$) in $t \approx 9 \times 10^{-8}s$(!). Or 1 gram in $\sigma = 10^{-6}m$ spreads to $2 \times 10^{-6}m$ in $1.6 \times 10^{19}s$(!). Uncertainty principle again.

• Compute expectation values using probabilities, e.g.

$$\langle f(x)\rangle = \int_{-\infty}^{\infty} f(x)|\psi(x,t)|^2 dx, \qquad \langle F(p)\rangle = \int_{-\infty}^{\infty} F(p)|\phi(p)|^2 dp.$$

Can always compute either in position or momentum space, e.g.

$$\langle f(x)\rangle = \int_{-\infty}^{\infty} \phi^*(p,t) f(i\hbar\frac{d}{dp})\phi(p,t)dp, \qquad \langle F(p)\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)F(-i\hbar\frac{d}{dx})\psi(x,t)dx,$$

where $\phi(p,t) = \phi(p)e^{-iE(p)t/\hbar}$. In position space, we replace $p \to -i\hbar\frac{d}{dx}$ and in momentum space we replace $x \to i\hbar\frac{d}{dp}$. This lead into the basic postulates of Q.M.

• Examples: use above gaussian wavefunction, $\langle p\rangle = p_0$ and $\langle x\rangle = p_0t/m$. Expectation values satisfy expected classical relations , example of Ehrenfest's theorem.