4/24/07 Lecture 7 outline

• Last time we wrote down the Schrodinger equation. Write it out again, for 3d:

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) = H\psi \equiv \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x})\right)\psi(\vec{x},t).$$

• Make some aside comments on some general consequences. First aside: particles don't disappear in QM. Indeed, probability density $\rho = |\psi(\vec{x},t)|^2$, and probability current density $\vec{J} = \frac{\hbar}{2mi}(\psi^*\nabla\psi - (\nabla\psi^*)\psi)$ satisfies a conservation equation, $\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0$ (since the potential V is real). This follows from the Schrodinger equation:

$$\frac{\partial\rho}{\partial t} = \frac{\partial\psi^*}{\partial t}\psi + \psi^*\frac{\partial\psi}{\partial t} = \frac{1}{i\hbar}\left(-(H\psi)^*\psi + \psi^* H\psi\right) = -\nabla\cdot\vec{J}.$$

• For general real potential V, can show (using Schrodinger equation)

$$\frac{d}{dt}\langle\vec{x}\rangle = \int d^3x\vec{x}\frac{\partial\rho}{\partial t} = \int d^3x\vec{J} = \frac{1}{m}\langle\vec{p}\rangle.$$

$$\frac{d}{dt}\langle\vec{p}\rangle = \frac{d}{dt}\int d^3x\psi^*\frac{\hbar}{i}\nabla\psi = -\int\psi^*\nabla V\psi = \langle\vec{F}\rangle.$$

Which is are examples of Ehrenfest's theorem: the expectation values satisfy the classical relations (modulo caveats below). More generally, for any observable u, the Schrodinger equation implies

$$\frac{d}{dt}\langle u\rangle = \langle\frac{\partial}{\partial t}u\rangle + \frac{1}{i\hbar}\langle[u,H]\rangle,$$

which is of the same form as the statements about classical mechanics, where the statement applies to the expectation values, and the Poisson brackets are replaced with commutators,

$$\{A,B\} \to \frac{1}{i\hbar}[A,B] \qquad \text{where} \qquad [A,B] \equiv AB - BA.$$

This is a general statement about quantum mechanics. In particular, get

$$\frac{d}{dt}\langle p\rangle = \langle -\frac{\partial V}{\partial x}\rangle$$

which is similar to the classical equation of motion, except that in general

$$\langle F(x)\rangle = \langle\frac{-\partial V}{\partial x}\rangle \neq F(\langle x\rangle) = -\frac{\partial}{\partial\langle x\rangle}V(\langle x\rangle).$$

This is approximately true (leading the Ehrenfest's theorem) only if the potential is slowly varying and Δx is not too big.

• In QM, we replace observables, like position, momentum, energy, angular momentum, etc. with linear operators. These operators act on the wavefunction. We have already seen this with $E \to i\hbar\frac{\partial}{\partial t}$, which according to the S.E. is $E \to H$. And $p \to -i\hbar\frac{\partial}{\partial x}$. The general rule is that the Poisson brackets of classical mechanics are replaced with commutators of the operators in quantum mechanics:

$$[A, B] = i\hbar\{A, B\},$$

where $[A, B] \equiv AB - BA$. In your HW, you will do some checks that this makes sense. In particular,

$$[x, p_x] = i\hbar, \quad [y, p_y] = i\hbar, \quad [z, p_z] = i\hbar.$$

The classical limit is $\hbar \to 0$, where the operators commute.

• When we measure the observable, we measure an eigenvalue of the corresponding operator. The measurement process alters the state of the system: the final state in the eigenvector corresponding to the measured eigenvalue.

• If two operators do not commute, such as position and momentum above, they both can not be measured simultaneously. Leads to the Heisenberg uncertainty principle, $\Delta x \Delta p_x \geq \hbar/2$.

• Example: free particle in a box, between $x = 0$ and $x = L$. Energy eigenvectors $u_n(x) = \sqrt{\frac{2}{L}}\sin(n\pi x/L)$, with eigenvalue $E_n = \hbar^2\pi^2 n^2/2mL^2$, for $n = 1, 2, 3, \ldots$. Note groundstate energy $E_1 \neq 0$, connect to uncertainty principle.