130a Homework 4, due 5/1

1. Shankar 1.8.10 (page 46).

2. Recall the definition of adjoint: if $A|\psi\rangle \equiv |A\psi\rangle$, then $\langle A\psi|\chi\rangle = \langle\psi|A^\dagger|\chi\rangle$, for any $|\psi\rangle$ and $|\chi\rangle$. Recall the definition of Hermitian: A is Hermitian $A = A^\dagger$.
   (a) Show that. if A and B are Hermitian, then so is $(A+B)^n$.
   (b) Show that, if A and B are Hermitian, then so is $i[A,B]$.
   (c) Use the definition above to show that $(AB)^\dagger = B^\dagger A^\dagger$.
   (d) Show that $A^\dagger A$ is Hermitian, even if A is not.

3. Consider a $K = 2$ dimensional, complex vector space, with orthonormal basis kets $|e_i\rangle$, for $i = 1, 2$. (For example, you can take $|e_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|e_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$). Consider the operator $A = e^{i\phi}|e_1\rangle\langle e_2| + e^{-i\phi}|e_2\rangle\langle e_1|$, where ϕ is an arbitrary real number.
   (a) Verify that $A^\dagger = A$.
   (b) Compute the eigenvalues λ_1 and λ_2 of A, and verify that they are real.
   (c) Find the corresponding eigen-kets $|\lambda_1 >$ and $|\lambda_2 >$, as linear combinations of the $|e_i\rangle$. Verify that the $|\lambda_i\rangle$ are orthogonal, and normalize them so that they are orthonormal, $< \lambda_i|\lambda_j >= \delta_{ij}$.
   (d) Using the above results, explicitly verify that $A = \sum_{i=1}^{2} \lambda_i |\lambda_i\rangle\langle\lambda_i|$.
   (e) Compute the expectation value of the operator A, defined to be $\langle A\rangle \equiv \langle\psi|A|\psi\rangle$, in the state $|\psi\rangle >\equiv a|e_1\rangle + b|e_2\rangle$, where a and b are arbitrary *complex* numbers. (Your answer for $\langle A\rangle$ should be real, since that's the case for any Hermitian operator, in any state.)

4. Shankar 1.10.4.

5. This problem is a simpler variant of 4.2.1 in Shankar. Consider the following three hermitian operators

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   These are operators on a $K = 2$ dimensional space.
   (a) What are the possible values one can measure if C is measured?
   (b) Take the state in which $C = 1$. In this state, what are the values of $\langle A\rangle$ and $\langle A^2\rangle$ and ΔA?
   (c) If the state is measured to have $C = 1$, and subsequently A is measured, what are the possible outcomes of the A measurement, and with what probabilities?

(d) Suppose that you first measures $C = 1$, what is the probability that an immediately subsequent measurement of C again in the state will yield $C \neq 1$?

(e) Suppose that you first measure $C = 1$, and then you measure that $A = 1$, what is the probability that, immediately after, you next measure C again and find $C \neq 1$?

(f) Compute $[A, C]$ and comment on how the answer fits with how your results for how parts (d) and (e) above compare.

6. Using $[\widehat{x}, \widehat{p}] = i\hbar$, verify

(a) that $\widehat{a} \equiv \sqrt{\frac{\alpha}{2\hbar}}\widehat{x} + i\sqrt{\frac{1}{2\alpha\hbar}}\widehat{p}$ satisfies $[\widehat{a}, \widehat{a}^\dagger] = 1$, for arbitrary real parameter α.

(b) Compute $a^\dagger a$ and show that it has three terms, one proportional to $\widehat{x}^2$, one to $\widehat{p}^2$, and a constant.