5/22/08 Homework 7 Due May 29, 2007.

1. Consider a quantum mechanical system which has only two available states, i.e. the ket $|\psi\rangle$ is a vector in a $K = 2$ dimensional, complex vector space. This space has a complete, orthonormal basis of kets $|e_i\rangle$, for $i = 1, 2$. (For example, you can take $|e_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|e_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$). The Hamiltonian of this system is $H = \alpha|e_1\rangle\langle e_2| + \alpha|e_2\rangle\langle e_1|$, where α is a positive real number. There is another observable, B, which is given by the operator $\widehat{B} = b_1|e_1\rangle\langle e_1| + b_2|e_2\rangle\langle e_2|$, where b_1 and b_2 are real numbers, with $b_2 > b_1$.

   The following experiments are performed in sequence:

   (# 1) At time $t_1 = 0$, the energy is measured.

   (# 2) At time t_2 (with $t_2 > 0$), the energy is measured again.

   (# 3) At time t_3 (with $t_3 > t_2$), the observable B is measured.

   (#4) At time t_4 (with $t_4 > t_3$), the energy is measured again.

   Answer the following questions about the outcomes of these experiments:

(a) Suppose that just before $t = 0$, the initial state is a completely general ket $|\psi\rangle$ (normalized, as always, by the condition that $\langle\psi|\psi\rangle = 1$). Write the possible outcomes of experiment (#1), and their probabilities, in terms of the numbers $\langle e_1|\psi\rangle$ and $\langle e_2|\psi\rangle$. (Your probabilities should add to 1.)

(b) Suppose that the outcome of experiment (# 1) turns out to be the larger of the two possibilities. Write the ket describing the state of the system immediately after the measurement at $t = 0$, in terms of a linear combination of the $|e_i\rangle$.

(c) Supposing the outcome mentioned above for experiment (# 1), what are the possible outcomes of experiment (# 2), and with what probabilities? Does your answer depend on what t_2 is?

(d) Suppose that the outcome of experiment (# 2) is the largest possibility for the energy. What are the possible outcomes of experiment (# 3), and with what probabilities? Does your answer depend on what t_3 is?

(e) Suppose that the outcome of experiment (# 3) is the largest of the possibilities. What is the ket describing the state of the system immediately after the measurement at time t_3?

(f) What are the possible outcomes of experiment (# 4), and with what probabilities? Write your answer as a function of t_4, if it depends on what t_4 is.

2. Consider the same system as above (but without the above sequence of experiments). Compute $[B, H]$. Suppose that the system is initially in a general state $|\psi(0)\rangle = |e_1\rangle\langle e_1|\psi(0)\rangle + |e_2\rangle\langle e_2|\psi(0)\rangle$. Write an expression for $\frac{d}{dt}\langle B\rangle$, and use it to compute $\langle B\rangle$ as a function of t, for general $t > 0$.

3. Consider the harmonic oscillator.

   (a) Write the equation $a|0\rangle = 0$ in the momentum basis, i.e. $\langle p|a|0\rangle = 0$. Integrate this differential equation to find $\phi_0(p) = \langle p|0\rangle$. Be sure to normalize $\phi_0(p)$ properly, so that $\langle 0|0\rangle = 1$.

   (b) Write the equation $|n\rangle = (n!)^{-1/2}(a^\dagger)^n|0\rangle$ in momentum space, as an expression for $\phi_n(p) = \langle p|n\rangle$. Just write the answer for $\phi_n(p)$ as a differential operator acting on $\phi_0(p)$.

4. Consider the harmonic oscillator, with initial condition $|\psi(t=0)\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$. Compute the expectation values $\langle x\rangle$ and $\langle p\rangle$ at a general later time t.

5. Using $\vec{L} = \vec{x} \times \vec{p}$, and the commutators $[x_a, p_b] = i\hbar\delta_{ab}$ (where a and b run over the x, y, and z components)

   (a) Write out L_x and L_y and L_z in terms of the components of $\vec{x}$ and $\vec{p}$. Verify that the angular momenta components are all Hermitian, e.g. $L_x^\dagger = L_x$.

   (b) Show that $[L_y, L_z] = i\hbar L_x$.

   (c) Compute $[x, L_x]$, $[y, L_x]$, $[z, L_x]$, and $[p_x, L_x]$, $[p_y, L_x]$, and $[p_z, L_x]$.

   (d) Compute $[\vec{x}\cdot\vec{x}, L_x]$.