5/29/08 Lecture 17 outline

• Last time note that different components of the angular momentum don't commute! Angular momentum commutation relations $[L_x, L_y] = i\hbar L_z$ and cyclic permutations. Implies that we can't measure these different components simultaneously.

• Convention is to diagonalize L_z. Show that $[L^2, L_z] = 0$, so can also diagonalize L^2. We can prove that the L^2 and L_z eigenvalues are quantized in units of $\hbar$, and labeled by integers ℓ and m:

$$L^2|\ell m\rangle = \hbar^2 \ell(\ell+1)|\ell m\rangle \qquad L_z|\ell m\rangle = \hbar m|\ell m\rangle$$

where

$$\ell = 0, 1, 2 \ldots \qquad \text{and} \qquad m = -\ell, -\ell+1, \ldots \ell,$$

so there are $2\ell+1$ choices for m. Note that the units work, since $\hbar$ has the units of angular momentum. Note that the maximum of L_z^2 is $\hbar^2\ell^2$, which is less than L^2; this is because $[L_x, L_y] = i\hbar L_z$ forbids setting $L_x^2 + L_y^2$ to zero for non-zero L_z.

• Here is how to prove the above. Recall that for any ket $|\psi\rangle$, the bra-ket $\langle\psi|\psi\rangle \equiv |||\psi\rangle||^2 \geq 0$, with $|||\psi\rangle||^2 \geq 0$ if and only if the ket vanishes, $|\psi\rangle = 0$. Moreover, for any operator A,

$$\langle A^\dagger A\rangle \equiv \langle\psi|A^\dagger A|\psi\rangle \equiv ||A|\psi\rangle||^2 \geq 0,$$

again with equality iff $A|\psi\rangle = 0$. In QM we have $L_a^\dagger = L_a$, as is the case for all physical observables. So we see from the above that $\langle L_x^2\rangle \geq 0$, and $\langle L_y^2\rangle \geq 0$ and $\langle L_z^2\rangle \geq 0$. To do better, introduce:

• Raising and lowering operators (analogous to creation and annihilation operators in SHO): $L_\pm \equiv L_x \pm iL_y$, satisfy $[L_z, L_\pm] = \pm\hbar L_\pm$ and $[L^2, L_\pm] = 0$. It follows that $L_\pm$ raises the L_z eigenvalue by $\pm\hbar$, but leaves alone the L^2 eigenvalue. Let's call $L^2|\alpha\beta\rangle = \alpha|\alpha\beta\rangle$ and $L_z|\alpha\beta\rangle = \beta|\alpha\beta\rangle$. We then get $L_\pm|\alpha\beta\rangle \sim |\alpha, \beta \pm \hbar\rangle$. The $L_\pm$ rotate the $\vec{L}$ vector to point more, or less, along the $\hat{z}$ axis.

• Note $L_\pm L_\mp = L^2 - L_z^2 \pm \hbar L_z$, and that $L_\pm^\dagger = L_\mp$. So $\langle L_\pm L_\mp\rangle \geq 0$ and $\langle L_\pm L_\mp\rangle \geq 0$ in any state. In particular, in the state $|\alpha\beta\rangle$ we have

$$\langle L_+ L_-\rangle = \alpha - \beta^2 + \hbar\beta \geq 0 \qquad \text{and} \qquad \langle L_- L_+\rangle = \alpha - \beta^2 - \hbar\beta \geq 0,$$

where we've fixed the normalization by $\langle\alpha, \beta|\alpha, \beta\rangle = 1$. Note that this also determines the normalization in $L_\pm|\alpha\beta\rangle \sim |\alpha, \beta \pm \hbar\rangle$:

$$L_\pm|\alpha, \beta\rangle = \sqrt{\alpha - \beta^2 \mp \hbar\beta}|\alpha, \beta \pm \hbar\rangle.$$

• But we saw that we can raise and lower β by acting on $|\alpha, \beta\rangle$ with $L_\pm$, which leaves α unchanged but takes $\beta \to \beta \pm \hbar$. If α and β were general numbers, by enough raising or lowering we'd eventually violate the above inequalities. The only way to avoid this is if there is a β_{max}, such that $L_+|\alpha, \beta_{max}\rangle = 0$, and a β_{min} such that $L_-|\alpha, \beta_{min}\rangle = 0$. It follows from the above then that $\alpha = \beta_{max}^2 + \beta_{max}\hbar = \beta_{min}^2 - \beta_{min}\hbar$. So $\beta_{min} = -\beta_{max}$. Moreover, must have that $L_-^N|\alpha, \beta_{max}\rangle \sim |\alpha, \beta_{max} - N\rangle$ must eventually vanish, so there is some integer N such that $\beta_{max} - N = \beta_{min}$, i.e. $2\beta_{max} = N$. So β_{max} can either be an integer or a half integer.

• For orbital angular momentum, $\beta_{max} \equiv \ell$ is an integer. Nature also use the half-integer possibility, in the context of spin: fermions have half-integer total angular momentum, given by $\vec{J} = \vec{L} + \vec{S}$, where $\vec{L}$ is the orbital part and $\vec{S}$ is the spin part. Ignore $\vec{S}$ for now, discuss it later.

• Instead of labeling the kets by α and β, label by ℓ and m, where $\alpha = \hbar^2\ell(\ell+1)$ and $\beta = \hbar m$, and m runs from ℓ to $-\ell$, in integer steps (so there are $2\ell + 1$ values of m):

$$L^2|\ell, m\rangle = \hbar^2\ell(\ell+1)|\ell, m\rangle \qquad L_z|\ell, m\rangle = m\hbar|\ell, m\rangle$$

and

$$L_\pm|\ell, m\rangle = \hbar\sqrt{\ell(\ell+1) - m^2 \mp m}|\ell, m \pm 1\rangle.$$

• The $|\ell, m\rangle$ form a complete, orthonormal basis:

$$\langle \ell', m'|\ell, m\rangle = \delta_{\ell,\ell'}\delta_{m,m'} \qquad \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} |\ell, m\rangle\langle \ell, m| = \mathbf{1}.$$

• Now consider these kets in position space.

Use spherical coordinates. The $|\ell, m\rangle$ states are independent of the radial coordinate, r; they depend only on θ and ϕ. To see why, write $\vec{L} = \vec{x} \times \vec{p}$ in position space, by replacing $\vec{p} \to -i\hbar\nabla$. Converting to spherical coordinates, get

$$L_z \to -i\hbar\frac{\partial}{\partial\phi} \qquad L_\pm \to \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

and

$$L^2 \to -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right].$$

In position space the L^2 and L_z eigenkets become $\langle\theta, \phi|\ell, m\rangle = Y_{\ell,m}(\theta, \phi)$. Their definition in terms of their eigenvalue equations, $L^2 Y_{\ell,m}(\theta, \phi) = \hbar^2\ell(\ell+1)Y_{\ell,m}(\theta, \phi)$ and $L_z Y_{\ell,m}(\theta, \phi) = m\hbar Y_{\ell,m}(\theta, \phi)$ are well known equations: the $Y_{\ell,m}(\theta, \phi)$ are the Spherical Harmonics, which always enter in solving problems in spherical coordinates.