4/17/09 Lecture outline

$\star$ Reading: Zwiebach chapter 4 and 5.

• Recall that $[S] = ML^2/T$, same as $[\hbar]$. (Indeed, Feynman's formulation of QM is based on $\psi \sim \int_{\text{all paths } x(t)} [dx(t)] e^{iS[x(t)]/\hbar}$.)

As mentioned last time, the action for a relativistic point particle of mass m is $S = -mc\int ds = -mc^2 \int dt\sqrt{1 - v^2/c^2}$. This gives $\vec{p} = \partial_{\vec{v}} = \gamma m\vec{v}$ and $H = \vec{p}\cdot\vec{v} - L = \gamma mc^2$, both of which are constants of the motion (thanks to the time and spatial translation invariance).

• Reparametrization invariance: write $x_\mu(\tau)$, and can change worldline parameter τ to an arbitrary new parameterization $\tau'(\tau)$, and the action is invariant. To see this use $S = -mc\int \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$ and change $\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\tau'}\frac{d\tau'}{d\tau}$ and note that $S \to S$. The Euler Lagrange equations of motion are $\frac{dp_\mu}{d\tau} = 0$.

When the particle is charged and in the presence of electric and magnetic fields, there is the new term in the action $S = -mc\int ds + \frac{q}{c}\int A_\mu dx^\mu$, which is also reparameterization invariant. The equations of motion can now be written as $\frac{d^2x^\mu}{d\tau^2} = \frac{q}{mc}F_{\mu\nu}\frac{dx^\nu}{d\tau}$.

• A key concept from last lecture is that E&M is associated with the local gauge transformation

$$\psi(t,\vec{x}) \to e^{iqf(t,\vec{x})/\hbar c}\psi(t,\vec{x}), \qquad A_\mu \to A_\mu + \partial_\mu f(t,\vec{x}). \tag{1}$$

According to Noether's theorem, there is a one-to-one correspondence

(continuous) global symmetry $\leftrightarrow$ conserved quantity.

The original example the relation between translation symmetry in time and/or space, $x^\mu \to x^\mu + a^\mu$, and conservation of energy and/or momentum, p^μ.

There is another deep correspondence

local gauge symmetry $\to$ forces.

and E&M is the force associated with the local symmetry above. There is still a conserved charge, in E&M it is current conservation $\partial^\mu j_\mu = 0$. As we'll now discuss, in general relativity (GR) the above spacetime translation symmetry is a subgroup of a more general symmetry, general coordinate invariance, which is the fundamental symmetry principle associated with gravity.

• A brief (!) introduction to general relativity. We replace the metric $\eta_{\mu\nu}$ with a dynamical quantity $g_{\mu\nu}$. There is a symmetry principle which is akin to the gauge invariance of electricity and magnetism and to the above repraraameterization invariance. It is general coordinate invariance: $x^\mu \to x^{\mu'}(x^\mu)$. Physics is invariant under such local coordinate changes. The metric transforms as $g_{\mu\nu} = g_{\mu'\nu'}\frac{dx^{\mu'}}{dx^\mu}\frac{dx^{\nu'}}{dx^\nu}$. The action of a point particle is $S = -mc\int ds + \frac{q}{c}\int A_\mu dx^\mu$, just like before, except that we contract and raise and lower indices with $g_{\mu\nu}$ rather than $\eta^{\mu\nu}$. Get from the Euler Lagrange equations now

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{q}{mc}F^\mu_\nu\frac{dx^\nu}{d\tau},$$

where

$$\Gamma^\mu_{\rho\sigma} = \tfrac{1}{2}g^{\mu\lambda}(\partial_\rho g_{\lambda\sigma} + \partial_\sigma g_{\lambda\rho} - \partial_\lambda g_{\rho\sigma})$$

is the connection; it is analogous to A^μ in electromagnetism. The connection enters into covariant derivatives like $\nabla_\rho V^\mu = \partial_\rho V^\mu + \Gamma^\mu_{\rho\sigma}V^\sigma$ in order to have things transform properly under general coordinate transformations (analogous to the gauge invariant covariant derivatives $D_\mu = \partial_\mu - i\frac{q}{\hbar c}A_\mu$. in E&M). The above equations of motion is called the geodesic equation; it reparameterization invariant ($\tau \to \tau'$) and transforms properly under general coordinate transformations $x^\mu \to x^{\mu'}$.

The Riemann tensor is

$$R^\rho_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - (\mu \leftrightarrow \nu).$$

It is analogous to $F_{\mu\nu}$ in E&M. The Ricci tensor is $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$ and the Ricci scalar is $R = R^\mu_\mu$. The metric is dynamically determined by minimizing the action w.r.t. $\delta g_{\mu\nu}$, where there is a term

$$S = \frac{1}{16\pi G_D}\int d^Dx\sqrt{|g|}R + \ldots.$$

For fun, we wrote it in general spacetime dimension D. Let's note the units (setting c=1): $[R] = L^{-2}$ and $[S] = ML$, so $[G_D] = L^{D-3}M^{-1}$. Since $[\hbar] = ML$, we have $G_D = \ell_P^{D-2}$ in D spacetime dimensions. (Note that $\int d^Dx\sqrt{|g|}$ gives the spacetime volume (which is clearly general coordinate invariant). This comment will be useful very soon, when we write down the relativistic string action!)

Note also that the relation $G_D = GV_C$ is evident from the above action.

In the weak curvature limit, we can reduce to the gravitational potentials, with $\nabla^2 V_g^{(D)} = 4\pi G_D\rho_m$. This comes from $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$ and $h_{0,0} \approx -2V_g$.

• Nonrelativistic strings. $[T_0] = [F] = [E]/L = [\mu_0][v^2]$. Indeed, considering $F = ma$ for an element dx of the string yields the string wave equation $\frac{\partial^2 y}{\partial x^2} - \frac{1}{v_0^2}\frac{\partial^2 y}{\partial t^2} = 0$, with $v_0 = \sqrt{T_0/\mu_0}$. Endpoints at $x = 0$ and $x = a$. Can choose Dirichlet or Neumann BCs at these points. With Dirichlet at each end, $y_n(x) = A_n \sin(n\pi x/a)$ and the general solution is $y(x,t) = \sum_n y_n(x) \cos\omega_n t$, where $\omega_n = v_0 n\pi/a$ (and the A_n are determined from the initial conditions, by Fourier transform).

The nonrelativistic string action is $S = \int dtL$ where L is the kinetic energy minus potential energy, which gives

$$S = \int dt \int dx \left(\tfrac{1}{2}\mu_0 (\frac{\partial y}{\partial t})^2 - \tfrac{1}{2} T_0 (\frac{\partial y}{\partial x})^2 \right),$$

which is a particular case of the more general action $S = \int dtdx \mathcal{L}(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x})$. We can then define the momentum density and corresponding quantity

$$\mathcal{P}^t = \frac{\partial \mathcal{L}}{\partial \dot{y}}, \qquad \mathcal{P}^x = \frac{\partial \mathcal{L}}{\partial y'},$$

and the action is made stationary, $\delta S = 0$, if

$$\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x} = 0,$$

which when applied to the above particular choice of action gives the usual wave equation. Note that Neumann or Dirichlet BCs correspond to $\mathcal{P}^x = 0$ or $\mathcal{P}^t = 0$ at the boundary, respectively, and that this is indeed needed for the surface terms to be compatible with $\delta S = 0$.