5/22/09 Lecture outline

$\star$ Reading: Zwiebach chapter 11 and 12.

• Finish from last time the gauge field in light cone gauge. Maxwell's equations in vacuum are $\partial_\nu F^{\mu\nu} = 0$, which with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ give $\partial^2 A^\mu - \partial^\mu(\partial \cdot A) = 0$. In momentum space, $p^2 A^\mu - p^\mu(p \cdot A) = 0$. The gauge transformation $A_\mu \to A_\mu + \partial_\mu f$ becomes $\delta A_\mu(p) = ip_\mu f(p)$. In light cone gauge, we use this freedom to set $A^+(p) = 0$. Taking $\mu = +$ in the EOM then implies $(p \cdot A) = 0$ (which implies that $A^- = (p^I A^I)/p^+$) and it then also follows that $p^2 A^\mu(p) = 0$. So we see that there are $D-2$ possible photon polarizations, with the 1-photon states given by $\sum_{I=2}^{D-1} \xi_I a^{I\dagger}_{p^+,\vec{p}_T}|\Omega\rangle$.

• Recall the relativistic point particle, with $S = \int L d\tau$ and $L = -m\sqrt{-\dot{x}^2}$, where $\dot{\equiv} \frac{d}{d\tau}$. ($\tau$ is taken to be dimensionless.) The momentum is $p_\mu = \partial L/\partial \dot{x}^\mu = m\dot{x}_\mu/\sqrt{-\dot{x}^2}$ and the EOM is $\dot{p}_\mu = 0$. In light cone gauge we take $x^+ = p^+\tau/m^2$. Then $p^+ = m\dot{x}^+/\sqrt{-\dot{x}^2}$ and the light cone gauge condition implies $\dot{x}^2 = -1/m^2$, so $p_\mu = m^2 \dot{x}_\mu$. Also, $p^2 + m^2 = 0$ yields $p^- = (p^I p^I + m^2)/2p^+$, which is solved for p^- and then $\dot{x}^- = p^-/m^2$ is integrated to $x^- = p^-\tau/m^2 + x_0^-$. Also, $x^I = x_0^I + p^I\tau/m^2$. The dynamical variables are (x^I, x_0^-, p^I, p^+).

• Heisenberg picture: put time dependence in the operators rather than the states, with $[q(t), p(t)] = i$ and

$$i\frac{d}{dt}\mathcal{O}(t) = i\frac{\partial \mathcal{O}}{\partial t} + [\mathcal{O}, H].$$

For time independent Hamiltonian, we have $|\psi(t)\rangle_S = e^{-iHt}|\psi\rangle_H$ and $\mathcal{O}_H = e^{iHt}\mathcal{O}_S e^{-iHt}$.

• Quantize the point particle in light cone gauge by taking the independent operators (x^I, x_0^-, p^I, p^+), with $[x^I, p^J] = i\eta^{IJ}$ and $[x_0^-, p^+] = i\eta^{-+} = -i$. These commutators are for either S or H picture, with the operators being functions of τ in the H picture. The remaining variables are defined by $x^+(\tau) = p^+\tau/m^2$, $x^-(\tau) = x_0^- + p^-\tau/m^2$, $p^- = (p^I p^I + m^2)/2p^+$ (the first two are explicitly τ dependent even in the S picture).

The Hamiltonian is $\sim p^-$, which generates $\frac{\partial}{\partial x^+}$ translations. Since $\frac{\partial}{\partial \tau} = \frac{p^+}{m^2}\frac{\partial}{\partial x^+} \leftrightarrow \frac{p^+}{m^2}p^-$ the Hamiltonian is

$$H = \frac{p^+ p^-}{m^2} = \frac{1}{2m^2}(p^I p^I + m^2).$$

Verify e.g.

$$i\frac{d}{d\tau}p^\mu = [p^\mu, H] = 0, \qquad i\frac{dx^I}{d\tau} = [x^I, H] = i\frac{p^I}{m^2},$$

reproducing the correct EOM. Likewise, verify $\dot{x}_0^- = 0$ and $\dot{x}^+ = \partial_\tau x^+ = p^+/m^2$. The momentum eigenstates are labeled by $|p^+, p^I\rangle$ and these are also energy eigenstates, $H|p^+, p^I\rangle = \frac{1}{2m^2}(p^I p^I + m^2)|p^+, p^I\rangle$.

• Connect the quantized point particle with the excitations of scalar field theory via

$$|p^+, p^I\rangle \leftrightarrow a^\dagger_{p^+,p^I}|\Omega\rangle.$$

The S.E. of the quantum point particle wavefunction maps to the classical scalar field equations, e.g. in light cone gauge:

$$(i\partial_\tau - \frac{1}{2m^2}(p^I p^I + m^2))\phi(\tau, p^+, p^I) = 0$$

is either the quantum S.E. of the point particle or the classical field equations of a scalar field.

• Lorentz transformations correspond to the inf. transformations $\delta x^\mu = \epsilon^{\mu\nu}x_\nu$, and the corresponding conserved Noether charges are the generalized angular momenta $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$. These generate rotations and boosts. Have e.g. $[M^{\mu\nu}, x^\rho] = i\eta^{\mu\rho}x^\nu - i\eta^{\nu\rho}x^\mu$ and $[M^{\mu\nu}, M^{\rho\sigma}] = i\eta^{\mu\rho}M^{\nu\sigma} \pm$(perms).

In light cone coordinates, have M^{IJ}, $M^{\pm I}$, M^{+-}, with e.g. $[M^{+-}, M^{+I}] = iM^{+I}$ and $[M^{-I}, M^{-J}] = 0$. There are some ordering issues for some of these, where operators which don't commute need to be replaced with symmetrized averages, e.g. $M^{+-} = -\frac{1}{2}(x_0^- p^+ + p^+ x_0^-)$ and $M^{-I} = x_0^- p^I - \frac{1}{2}(x_0^I p^- + p^- x_0^I)$.

• Open string. Imposed constraints $(\dot{X} \pm X')^2 = 0$ to get

$$\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'}X^{\mu'}, \qquad \mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu.$$

In light cone gauge, much as with the point particle, the independent variables are $(X^I, (\sigma)x_0^-, \mathcal{P}^{\tau I}(\sigma), p^+)$. In the H picture the capitalized ones depend (implicitly) on τ too. The commutation relations are

$$[X^I(\sigma), \mathcal{P}^{\tau J}(\sigma')] = i\eta^{IJ}\delta(\sigma - \sigma'), \qquad [x_0^-, p^+] = -i.$$

The Hamiltonian is taken to be

$$H = 2\alpha' p^+ p^- = 2\alpha' p^+ \int_0^\pi d\sigma \mathcal{P}^{\tau -} = \pi\alpha' \int_0^\pi d\sigma (\mathcal{P}^{\tau I}\mathcal{P}^{\tau I} + X^{I'}X^{I'}(2\pi\alpha')^{-2})$$

Can write $H = L_0^\perp$ since $L_0^\perp = 2\alpha' p^+ p^-$.

This H properly yields the expected time derivatives, e.g. $\dot{X}^I = 2\pi\alpha'\mathcal{P}^{\tau I}$.

Recall the solution with N BCs:

$$X^I(\tau,\sigma) = x_0^I + \sqrt{2\alpha'}\alpha_0^I\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^I \cos n\sigma e^{-in\tau}. \tag{1}$$

The needed commutators are ensured by

$$[\alpha_m^I, \alpha_n^J] = m\eta^{IJ}\delta_{n+m,0}.$$

Also, as before, we define $\alpha_0^I \equiv \sqrt{2\alpha'}p^I$. Now define $\alpha_{n>0}^\mu = \sqrt{n}a_n^\mu$ and $\alpha_{-n}^\mu = a_n^{\mu *}\sqrt{n}$ to rewrite the above as

$$[a_m^I, a^{J^\dagger}] = \delta_{m,n}\eta^{IJ}. \tag{2}$$

- The transverse light cone coordinates can be described by

$$S_{l.c.} = \int d\tau d\sigma \frac{1}{4\pi\alpha'}(\dot{X}^I\dot{X}^I - X^{I'}X^{I'}).$$

Gives correct $\mathcal{P}^{\tau I} = \partial\mathcal{L}/\partial\dot{X}^I$ and correct $H = \int d\sigma(\mathcal{P}^{\tau I}\dot{X}^I - \mathcal{L})$.

Writing $X^I(\tau,\sigma) = q^I(\tau) + 2\sqrt{\alpha'}\sum_{n=1}^\infty q_N^I(\tau)n^{-1/2}\cos n\sigma$ and plugging into the action above gives

$$S = \int d\tau[\frac{1}{4\alpha'}\dot{q}^I\dot{q}^I + \sum_{n=1}^\infty(\frac{1}{2n}\dot{q}_n^I\dot{q}_n^I - \frac{n}{2}q_n^I q_n^I)]$$

and

$$H = \alpha' p^I p^I + \sum_{n=1}^\infty \frac{n}{2}(p_n^I p_n^I + q_n^I q_m^I).$$

A bunch of harmonic oscillators. Relate to (1) and (2), showing that the a_m can be interpreted as the usual harmonic oscillator annihilation operators.

- $X^+(\tau\sigma) = 2\alpha' p^+\tau = \sqrt{2\alpha'}\alpha_0^+\tau$. For X^- recall expansion, with $\sqrt{2\alpha'}\alpha_n^- = \frac{1}{p^+}L_n^\perp$, where $L_n^\perp \equiv \frac{1}{2}\sum_p \alpha_{n-p}^I\alpha_p^I$ is the transverse Virasoro operator. There is an ordering ambiguity here, only for $L_0^\perp$:

$$L_0^\perp = \tfrac{1}{2}\alpha_0\alpha_0 + \tfrac{1}{2}\sum_{p=1}^\infty \alpha_{-p}^I\alpha_p^I + \tfrac{1}{2}\sum_{p=1}^\infty \alpha_p^I\alpha_{-p}^I.$$

Ended here. Finish next time:

The ordering in the last terms need to be fixed, so the annihilation operator α_p is on the right, using $\alpha_p^I\alpha_{-p}^I = \alpha_{-p}^I\alpha_p^I + [\alpha_p^I, \alpha_{-p}^I]$, which gives

$$L_0^\perp = \alpha' p^I p^I + \sum_{n=1}^\infty n a_n^{I\dagger}a_n^I,$$

where the normal ordering constant has been put into

$$2\alpha' p^- = \frac{1}{p^+}(L_0^\perp + a), \qquad a = \tfrac{1}{2}(D-2)\sum_{p=1}^{\infty} p.$$

This leads to

$$M^2 = \frac{1}{\alpha'}(a + \sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I).$$

The divergent sum for a is regulated by using $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and analytically continuing to get $\zeta(-1) = -1/12$. So

$$a = -\frac{1}{24}(D-2).$$

• Virasoro generators and algebra (corresponds to worldsheet energy-momentum tensor). Since $(\alpha_n^I)^\dagger = \alpha_{-n}^I$, get $L_n^{\perp\dagger} = L_{-n}^\perp$. Also,

$$[L_m^\perp, \alpha_n^I] = -n\alpha_{n+m}^I.$$

$$[L_m^\perp, L_n^\perp] = (m-n)L_{n+m}^\perp + \frac{D-2}{12}(m^3 - m)\delta_{m+n,0}.$$

• Spacetime Lorentz symmetry corresponds to conserved currents on worldsheet, with conserved charges

$$M_{\mu\nu} = \int_0^\pi (X_\mu \mathcal{P}_\nu^\tau - (\mu \leftrightarrow \nu))d\sigma.$$

Plug in $\mathcal{P}_\nu^\tau = \frac{1}{2\alpha'}\dot{X}^\mu$ and plug in oscillator expansion of X^μ to get

$$M^{\mu\nu} = x_0^\mu p^\nu - x_0^\nu p^\mu - i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu).$$

In light cone gauge, have to be careful with M^{-I}, since X^- is constrained to something complicated,

$$X^-(\tau,\sigma) = x_0^- + \sqrt{2\alpha'}\alpha_0^- \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in\tau}\cos n\sigma, \qquad \sqrt{2\alpha'}\alpha_n^- = \frac{1}{p^+}L_n^\perp.$$

and also careful to ensure that $[M^{-I}, M^{-J}] = 0$. Find, after appropriately ordering terms,

$$M^{-I} = x_0^- p^I - \frac{1}{4\alpha' p^+}\left(x_0^I(L_0^\perp + a) + (L_0^\perp + a)x_0^I\right) - \frac{i}{\sqrt{2\alpha'}p^+}\sum_{n=1}^{\infty}\frac{1}{n}(L_{-n}^\perp \alpha_n^I - \alpha_{-n}^I L_n^\perp).$$

Then get

$$[M^{-I}, M^{-J}] = -\frac{1}{\alpha' p^{+2}} \sum_{m=1}^{\infty} (\alpha_{-m}^{I} \alpha_{m}^{J} - (I \leftrightarrow J))[m(1-((D-2)/24)) + m^{-1}(((D-2)/24)+a)].$$

Since this must be zero, get $D = 26$ and $a = -1$.

The worldsheet Hamiltonian is thus

$$H = 2\alpha' p^{+} p^{-} = L_{0}^{\perp} - 1.$$

- The states are obtained as

$$|\lambda\rangle = \prod_{n=1}^{\infty} \prod_{I=2}^{25} (a_{n}^{I\dagger})^{\lambda_{n,I}} |p^{+}, \vec{p}_{T}\rangle.$$

These states are eigenstates of

$$M^{2} = \frac{1}{\alpha'}(-1 + N^{\perp}), \qquad N^{\perp} \equiv \sum_{n=1}^{\infty} n a_{n}^{I\dagger} a_{n}^{I},$$

with eigenvalue

$$M^{2} = -1 + \sum_{n} \sum_{I} n \lambda_{n,I}.$$

The groundstate is tachyonic (!). The first excited state is a massless spacetime vector with $D - 2$ polarizations, i.e. a massless gauge field, like the photon (but in $D = 26$)!

The tachyon is related to the fact that the D25 brane is unstable, it decays to the closed string vacuum. The closed bosonic string is also unstable, as we'll see next time. These instabilities can be cured by adding fermions and considering the superstring. Then the critical spacetime dimension is $D = 10$.