Physics 220, Lecture 12

⋆ Reference: Georgi chapters 3-6.

• Last time: Clebsch-Gordon coefficients:

$$|jm, j_1 j_2\rangle = \sum_{m_1} \sum_{m_2} |j_1 m_1, j_2 m_2\rangle\langle j_1 m_1, j_2 m_2|jm, j_1 j_2\rangle.$$

Examples ($\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$, $\frac{1}{2} \otimes 1 = \frac{3}{2} \oplus \frac{1}{2}$.).

• The fundamental ($j = \frac{1}{2}$) of $SU(2)$ acts on vectors u_α, with $\alpha = 1, 2$: $u \to gu$. Upper and lower indices: upper index transforms as $\tilde{u} \to \tilde{u} g^\dagger$, such that δ_α^β transforms as $1 \to g 1 g^\dagger = 1$ since the matrices are unitary. Can also consider the complex conjugate representation, acting on v^α. The difference between a rep and its conjugate is is $T_a \to -T_a^*$, which satisfy the same Lie algebra. For $SU(2)$ these are isomorphic. $\epsilon_{\alpha\beta}$ is an invariant tensor. More general tensors $u_{\alpha_1 \ldots \alpha_\ell}$. The spin j representation corresponds to the tensor $u_{\alpha_1 \ldots \alpha_j}$, where the indices are symmetrized. For example, the $j = 1$ state is $X_\beta^\alpha = \sigma_i x^i$, which transforms as $X \to UXU^{-1}$. Since $\vec{x} \cdot \vec{x} = -\det X$, the invariance follows immediately. This map demonstrates how $SU(2)$ is a double cover of the $SO(3)$ rotation group: the $SU(2)$ elements U and $-U$ correspond to the same rotation.

Consider tensor $u^{i_1 \ldots i_{2j}}$, completely symmetrized, corresponding to the spin j representation. It has $J_3 = m$ if there are $j + m$ indices equal to 1, and $j - m$ indices equal to 2. There are $\begin{pmatrix} 2j \\ j+m \end{pmatrix}$ ways to do this. So write $|jm\rangle = \begin{pmatrix} 2j \\ j+m \end{pmatrix}^{-1/2} |v_{j,m}\rangle$, where $|v_{jm}\rangle$ is a bunch of Kronecker deltas. Application: use to derive the Clebsch $\langle j_1 + j_2, m_1 + m_2 | j_1 m_1; j_2 m_2\rangle$.