Physics 220, Lecture 15

$\star$ Reference: Georgi chapters 6-8

• Recall example of $SU(3)$ fundamental rep, $T_a = \frac{1}{2}\lambda_a$, Gell-Mann matrices, with $\lambda_3 = diag(1,-1,0)$ and $\lambda_8 = \frac{1}{\sqrt{3}}(1,1,-2)$. (Normalize to $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$. Since a rep and conjugate rep have $T_a$ and $-T_a^*$ respectively, note $\mathbf{3}$ and $\overline{\mathbf{3}}$ differ. $\lambda_{1,2}$ are Pauli matrices $\sigma_{1,2}$ in the $(1,2)$ components, $\lambda_{4,5}$ are $\sigma_{1,2}$ in the $(1,3)$ entries, and $\lambda_{6,7}$ are similar in the $(2,3)$ entries. So $E_{\pm 1,0} = \frac{1}{\sqrt{2}}(T_1 \pm iT_2)$, $E_{\pm 1/2, \pm\sqrt{3}/2} = \frac{1}{\sqrt{2}}(T_4 \pm iT_5)$, $E_{\mp 1/2, \pm\sqrt{3}/2} = \frac{1}{\sqrt{2}}(T_6 \pm iT_7)$.

• Last time: by considering the adjoint representation, write all the generators as the Cartan $H_i$, $i = 1\dots r =$rank$(G)$, and generators labeled by root vectors $\alpha$, such that $[H_i, E_\alpha] = \alpha_i E_\alpha$, $[E_\alpha, E_{-\alpha}] = \alpha\cdot H$.

• Lots of $SU(2)$s: write $E^\pm = |\alpha|^{-1}E_{\pm\alpha}$ and $E_3 = |\alpha|^{-2}\alpha\cdot H$ to get an $SU(2)$ subalgebra.

• Use to argue that if $\alpha$ is a root then $k\alpha$ is only a root if $k = 0, 1, -1$, and that there can only be a single root with any non-zero weight $\alpha$: another such root would have $E_3|E'_\alpha\rangle = |E'_\alpha\rangle$, but $0 = \langle E_\alpha|E'_\alpha\rangle$ implies $E^-|E'_\alpha\rangle = 0$, so lowest weight, but on the other hand $E_3|E'_\alpha\rangle = |E'_\alpha\rangle$ so like $J_3 = 1$. Impossible to have $J_3 = 1$ lowest weight state. Likewise, $k\alpha$ for any $k$ integer or half integer is similarly impossible.

Also, for any rep, $E_3|\mu, D\rangle = \frac{\alpha\cdot\mu}{\alpha\cdot\alpha}|\mu, D\rangle$, so $\frac{2\alpha\cdot\mu}{\alpha^2}$ is an integer. Can raise such a state using $E^+$ at most some integer $p$ times before getting zero, and can lower using $E^-$ at most some integer $q$ times before getting zero. Implies:

$$\frac{\alpha\cdot(\mu+p\alpha)}{\alpha^2} = \frac{\alpha\cdot\mu}{\alpha^2} + p = j, \qquad \frac{\alpha\cdot\mu}{\alpha^2} - q = -j.$$

So

$$\frac{\alpha\cdot\mu}{\alpha^2} = -\tfrac{1}{2}(p-q). \tag{1}$$

Applying this to the adjoint representation for any two roots $\alpha$ and $\beta$, conclude that

$$\cos^2\theta_{\alpha\beta} = \frac{(\alpha\cdot\beta)^2}{\alpha^2\beta^2} = \frac{(p-q)(p'-q')}{4}, \qquad = 0,\ \frac{1}{4},\ \tfrac{1}{2},\ \frac{3}{4}.$$

and then $\theta_{\alpha\beta}$ is either $\pi/2$, $\pi/6$ or $2\pi/6$, $\pi/4$ or $3\pi/4$, $\pi/6$ or $5\pi/6$.

• Highest weights: Chose a basis for the Cartan and call a weight positive or negative based on the sign of the first non-zero entry. Clearly arbitrary, but fine. Now say $\mu > \nu$ if $\mu - \nu$ is positive, and find one of the weights is highest in this ordering $\mu_{max}$. In adjoint, the positive roots are raising operators and the negative roots are lowering, so the raising operators must annihilate $\mu_{max}$. Then can fill out the irrep by applying lowering operators to $\mu_{max}$.

• Simple roots: positive roots that can't be written as sums of other positive roots. If a weight is annihilated by all simple roots, it is $\mu_{max}$.

If $\alpha$ and $\beta$ are simple roots, then $\alpha-\beta$ isn't a root. E.g. if $\beta-\alpha>0$ then $\alpha+(\beta-\alpha)=\beta$ would violate the condition that $\beta$ can't be a sum of two positive roots.

Therefore, $E_{-\alpha}|E_\beta\rangle = E_{-\beta}|E_\alpha\rangle = 0$ so $q=q'=0$ in (1):

$$\frac{\alpha\cdot\beta}{\alpha^2} = -\frac{p}{2}, \qquad \frac{\beta\cdot\alpha}{\beta^2} = -\frac{p'}{2}.$$

So $\beta^2/\alpha^2 = p/p'$ and $\cos\theta_{\alpha\beta} = -\sqrt{pp'}/2$, which implies $\frac{\pi}{2} \le \theta_{\alpha\beta} < \pi$.

Implies that the simple roots are linearly independent: $\gamma = \sum_\alpha x_\alpha \alpha$ can't vanish unless all $x_\alpha = 0$. To try to get $\gamma = 0$, would need some $x_\alpha$ positive and some negative, i.e. $\gamma = \mu - \nu$ where $\mu$ and $\nu$ each have all positive coefficients. But then $\gamma^2 > 0$ since all simple roots have $\alpha\cdot\beta \le 0$.

The simple roots are linearly independent and also complete: the number of them equals the rank of the group. They form a basis for all roots. If not, there would be a vector $\xi$ orthogonal to all the simple roots and that would imply that it's orthogonal to all roots and then that $[\xi\cdot H, \phi] = 0$, showing that $\xi\cdot H$ commutes with everything, violating the assumption that our algebra is simple.

Any positive root is of the form $\phi_k = \sum_\alpha k_\alpha \alpha$ where $\alpha$ runs over the simple roots and all $k_\alpha > 0$. Here $k = \sum_\alpha k_\alpha$. So $\phi_1$ are the simple roots. Given a $\phi_\ell$, consider $E_\alpha|\phi_\ell\rangle$. Since $2\alpha\cdot\phi_\ell = -\alpha^2(p-q)$, can determine $q$ and check sign of $p$; $\phi_\ell+\alpha$ is a root iff $p>0$. E.g. take $\phi_1 = \beta$ a simple root and then note that if $\alpha\cdot\beta = 0$ then $\alpha+\beta$ is not a root, whereas if $\alpha\cdot\beta = -p\alpha^2/2$ with $p\neq 0$, then $p>0$ and $\alpha+\beta$ is a root, $\alpha+\beta = \phi_2$.

For $SU(3)$, the simple roots satisfy $\alpha_1^2 = \alpha_2^2 = 1$, and $\alpha_1\cdot\alpha_2 = -\frac{1}{2}$. Follows that $\alpha_1+\alpha_2$ is a root, but adding any more $\alpha_1$ or $\alpha_2$s gives something that's not a root.

• Dynkin diagrams: each simple root is written as a node, with 0,1,2,3 lines connecting nodes if the angle is 90, 120, 135, 150° respectively.

• Under the $SU(2)$ associated with $\alpha_i$, any weight has $E_3|\mu\rangle = \frac{2H\cdot\alpha_i}{\alpha_i^2}|\mu\rangle$ and the eigenvalue is $2\mu\cdot\alpha_i/\alpha_i^2 = q_i - p_i$. Can use $q_i - p_i$ to label the weights.

The Cartan matrix for the simple roots is

$$A_{ji} \equiv 2\frac{\alpha_i\cdot\alpha_j}{\alpha_i^2}$$

The diagonal entries are 2 and the off-diagonals are $-p$.

The $j$-th row give the $q_i - p_i$ values of the simple root $\alpha_i$. Writing a positive root as $\phi = \sum_j \alpha_j$, it has $q_i - p_i = \sum_j k_j A_{ji}$. Raising $\phi \to \phi + \alpha_j$ shifts $q_i - p_i \to q_i - p_i + A_{ji}$. At $k = 0$ have the Cartan, with $E_3 = q_i - p_i = 0$. At $k = 1$ have the simple roots. Continue raising / lowering until the diagram is complete.

Example: $SU(3)$ , with $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, and $G_2$ with $A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$. $G_2$ has $\alpha_1 = (0, 1)$ and $\alpha_2 = (\sqrt{3}/2, -3/2)$ with $\alpha_1^2 = 1$ and $\alpha_2^2 = 3$ and $\alpha_1 \cdot \alpha_2 = -3/2$, so the angle between them is $150°$. The raising operators are $E_1^+ = E_{\alpha_1}$ and $E_2^+ = \frac{1}{\sqrt{3}} E_{\alpha_2}$. The positive roots are: $\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2$, $3\alpha_1 + \alpha_2$, $3\alpha_1 + 2\alpha_2$.

Can construct algebra, e.g. $|[E_{\alpha_1}, E_{\alpha_2}\rangle = \sqrt{\frac{3}{2}} E_{\alpha_1 + \alpha_2}\rangle$ etc.