Physics 220, Lecture 16

⋆ Reference: Georgi chapters 8-9, a bit of 20.

• Continue with Dynkin diagrams and the Cartan matrix,

$$A_{ji} \equiv 2\frac{\alpha_i \cdot \alpha_j}{\alpha_i^2}.$$

The j-th row give the $q_i - p_i = -p_i$ values of the simple root α_i's $SU(2)_i$ generators acting on the root α_j. Again, we always have

$$2\frac{\alpha_i \cdot \mu}{\alpha_i^2} = q_i - p_i, \tag{1}$$

where p_i and q_i are the number of times that the weight μ can be raised by E_{α_i}, or lowered by $E_{-\alpha_i}$, respectively, before getting zero. Applied to $\mu = \alpha_j$, we know that $q_i = 0$, since $E_{-\alpha_j}|\alpha_i\rangle = 0$, since $\alpha_i - \alpha_j$ is not a root for $i \neq j$.

Again, we then have $A_{ji}A_{ij} = pp' = 4\cos^2\theta_{ij}$, which must equal 0,1,2, or 3; these correspond to $\theta_{ij} = \pi/2$, $2\pi/3$, $3\pi/4$, and $5\pi/6$, respectively.

The Dynkin diagram has a node for each simple root (so the number of nodes is r =rank(G)), and nodes i and j are connected by $A_{ji}A_{ij}$ lines. When $\alpha_i^2 \neq \alpha_j^2$, sometimes it's useful to darken the node for the smaller root.

• Another example: constructing the roots for C_3, starting from $\alpha_1^2 = \alpha_2^2 = 1$, and $\alpha_3^2 = 2$, i.e. the Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}.$$

Find 9 positive roots.

• Classify all simple, compact Lie algebras from their A_{ji}. Require 3 properties: (1) $\det A \neq 0$ (since the simple roots are linearly independent); (2) $A_{ji} < 0$ for $i \neq j$; (3) $A_{ij}A_{ji}$ = 0,1, 2, 3. From these can prove many constraints. For example, taking $\alpha \equiv \sum_i \alpha_i/|\alpha_i|$ it follows from $\alpha^2 > 0$ that the number of joined nodes is strictly less than the number of nodes. Therefore, there can't be any closed loops. Draw all 2 allowed Dynkin diagrams for 3 nodes, following from fact that the sum of angles between any 3 linearly independent vectors must be $< 2\pi$. This same constraint applies to any subdiagram of a Dynkin diagrams. Another result following from this is that at most 3 lines can connect to any node. We can freely cut apart diagrams and shrink lines to get subsystems, which must satisfy same constraints. If a node γ connects with two single lines to two other nodes, α and β (with $\alpha \cdot \beta = 0$), then there is another allowed diagram where γ connects with a double node to $\alpha + \beta$. Draw several examples with non-linearly independent α_i, and corresponding μ_j such that $(\sum_j \mu_j \alpha_j)^2 = 0$ (note each μ_j is the average of those at connecting nodes). Draw all allowed Dynkin diagrams.

Extended Dynkin diagrams: include α_0, which is the lowest root, as an extra node in the diagram. Since it's lowest, $\alpha_0 - \alpha_j$ is not a root, so its $q = 0$ and thus A_{0j} and A_{j0} are non-positive integers. So satisfy same diagram rules as above, with the exception that there is a single linear relation among the $r + 1$ nodes – only r are linearly independent. The extended Dynkin diagram $\tilde{A}$ has rank r so $\det \tilde{A} = 0$. Draw the extended Dynkin diagrams, with the corresponding μ_J. The linear relation then says how the highest root is written in terms of the other roots.