Physics 220, Lecture 4

• Schur's lemmas.

First: suppose irreps D_1 acting on linear vector space L_1, and D_2 on L_2. Suppose there is a map A from L_1 into L_2 such that, for all $g \in G$, $D_2(G)AL_1 = AD_1(G)L_1 \subset AL_1 \subset L_2$, so $AD_1 = D_2A$ (intertwining), and then AL_1 is an invariant subspace of L_2. Since D_2 is supposed to be an irrep, either $AL_1 = L_2$, or $A = 0$.

Second: Let A map $L \to L$, and $AD(R) = D(R)A$. For a finite irrep $D(R)$, must have $A \propto 1$. A proof: consider an eigenvector, $Ax = \lambda x$, and space $X_\lambda = \{x|Ax = \lambda x\} \supset L$. Then show $D(R)X_\lambda \subset X_\lambda \subset L$, i.e. X_λ is an invariant subspace. Since an irrep, must have $X_\lambda = L$, i.e. $A = \lambda 1$.

• Power of Schur's lemmas in QM. Suppose operator O respects symmetry group G, which maps states $|\mu\rangle \to D(g)|\mu$. Condition that O matrix elements are unchanged by the symmetry operation on the states is equivalent to $[O, D(g)] = 0$ for all $g \in G$. Consider a rep $|a, j, x\rangle$, where a labels the irrep, $j = 1 \dots n_1$ label the irrep basis elements, and x are other labels. For example, for the hydrogen atom the rotation symmetry gives $|\ell, m, n\rangle$. Using Schur, show

$$\langle a, j, x|O|b, k, y\rangle = f_a(x, y)\delta_{ab}\delta_{jk}.$$

All the group theory label dependence is completely fixed by the symmetry. This is an example of a Wigner Eckart theorem, to be discussed in more detail later, together with applications in quantum mechanics.

• Orthogonality. Use Schur's lemmas to show

$$A^{ab}_{jl} \equiv \sum_{g\in G} D_a(g^{-1})|a, j\rangle\langle b, l|D_b(g) = \frac{|G|}{|r_a|}\delta_{ab}\delta_{jl}I.$$

Implies that $\sqrt{\frac{|r_a|}{|G|}}[D(g)]_{jk}$ are orthonormal functions of the group elements g. Implies $|G| \geq \sum_a |r_a|^2$. So for $|G| < \infty$ there is a finite set of irreps, of bounded dimension.

Consider the regular representation to argue that the above functions are also a complete set of functions of the group elements: every function of the group can be written in terms of the regular representation:

$$F(g) = \langle F|g\rangle = \langle FD_R(g)|e\rangle_R = \sum_{g'\in G} F(g')[D_R(g)]_{g'e}$$

and thus in terms of the irreps (since the regular rep is completely reducible). Thus $|G| = \sum_a |r_a|^2$.

Example: cyclic group Z_N and Fourier series: $D_n(a_j) = e^{2\pi i n j/N}$.

• Characters. $\chi_D(g) = \mathrm{Tr} D(g)$, invariant under similarity transformations, so independent of choice of basis of L, good.

Note that $\chi_D(g)$ only depends on the conjugacy class of g: $\chi(g) = \chi(g_1 g g_1^{-1})$. The characters are class functions.

Using the previous orthogonality relation, show that the characters satisfy an orthogonality relation:

$$\frac{1}{|G|} \sum_{g \in G} \chi_{D_a}(g)^* \chi_{D_b}(g) = \delta_{ab}.$$

Implies (number if irreps)$\leq$ number of conjugacy classes. In fact, they're equal:

The characters provide a complete set of class functions: any function of only conjugacy classes can be expanded in terms of the characters. As we already said,

$$F(g_1) = \sum_{a,j,k} c^a_{jk} [D_a(g_1)]_{jk}$$

Now on both sides replace g_1 with $g^{-1} g_1 g$ and average over all g. Use the orthogonality relations to do the g sum and get finally

$$F(g_1) = \sum_{a,j} \frac{1}{|r_a|} c^a_{jj} \chi_a(g_1).$$

So there is another orthogonality condition, involving the sum over irreps:

$$\sum_a \chi_{D_a}(g_\alpha)^* \chi_{D_a}(g_\beta) = \frac{|G|}{k_\alpha} \delta_{\alpha\beta},$$

where α and β label the conjugacy classes, and k_α is the number of elements in that class.