Physics 225, Homework 1, Due Monday April 4.

1. Relative to frame 1, frame 2 is boosted along the $+x$ axis by velocity v_{12}. Relative to frame 2, frame 3 is boosted along the $+x$ axis by velocity v_{23}. Find the Lorentz transformation between frames 1 and 3, **by matrix multiplying** those associated with the above two boosts, verifying that it corresponds to a boost of some relative velocity v_{13} between frames 1 and 3. Verify that this v_{13}, in terms of v_{12} and v_{23}, agrees with what you'd expect, based on the relativistic addition of velocities.

2. A photon in the lab frame has frequency ω and moves at an angle θ with respect to the x-axis. A rocket moves with speed v relative to the lab frame, along the $+x$ axis. Find the frequency ω' and the angle θ' of the photon according to an observer in the rocket frame.

3. The world line of a particle is described by the parametric equations in some Lorentz frame:
$$t(\lambda) = b\sinh(\lambda/b), \qquad x(\lambda) = b\cosh(\lambda/b).$$
b is a constant, and λ is a parameter. Compute the particle's four-velocity and four-acceleration components. Show that λ is the proper time along the world line. Show that the particle's acceleration (in its instantaneous rest frame) is uniform. Interpret b.

4. This is a **modified** version of Carroll 1.1. Take the rocket to move with velocity v along the x axis, with the back end of the rocket, at $x' = 0$, at position $x = vt$ in the lab frame. Someone in the rocket bounces a ball off of the wall (at $x' = 0$) at the back end of the rocket, with initial and final 3-velocities $\vec{v}'_i = (-V'\cos\theta', V'\sin\theta', 0)$ and $\vec{v}'_f = (V'\cos\theta', V'\sin\theta', 0)$. (Be careful: v is the velocity of the rocket relative to the lab frame, and V' is the velocity of the ball, according to an observer in the rocket frame.) Find the initial and final 3-velocities of the ball as observed by someone in the lab frame.

5. This is a **modified** version of Carroll 1.5. Muons have mass $m = 0.106GeV$ $(c = 1)$ and rest frame lifetime $2.16 \times 10^{-6}s$. Now consider a muon traveling in a straight line, at constant velocity, and suppose that its total energy measured by someone in the lab frame is 1000GeV. How long does it live, as seen by an observer in the lab frame? How far does it travel before decay, as seen by someone in the lab frame? What is this length, as measured in the muon's rest frame?

6. Using the transformation law applied to $F_{\mu\nu}$, show how $\vec{E}$ and $\vec{B}$ transform under a
   (a) Rotation around the y axis.
   (b) Boost along the z axis.

7. Verify that $T^{\mu\nu}_{EM\ field} = F^{\mu\lambda}F^{\nu}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}$ is conserved, using Maxwell's equations (in relativistic form). You can check it without source terms [1].

8. A rocket accelerates by ejecting part of its rest mass as exhaust. The speed of the exhaust is a constant value u in the rocket's rest frame. Use conservation of energy and momentum to find the ratio of final to initial rest mass for a rocket that accelerates from rest to a speed V. Hint: rest mass is not conserved – energy and momentum are conserved.

9. Consider a hypothetical particle, called a *tachyon*, that moves faster than the velocity of light.

(a) Consider the tangent 4-vector to the trajectory of a tachyon, written as $u^\mu = dx^\mu/ds$, where ds is the spacelike interval along the trajectory. Show that it satisfies $u_\mu u^\mu = +1$, and evaluate its components in terms of the 3-velocity $\vec{v} = d\vec{x}/dt$.

(b) Show that if the tachyon moves faster than light in one inertial frame, it also moves faster than light in any Lorentz boosted frame.

(c) Define 4-momentum by $p^\mu = mu^\mu$ and find the relation between energy and 3-momentum for a tachyon.

(d) Show that there is an inertial frame where the energy of a tachyon is negative.

(e) Show that the process $n \to n + t$ is compatible with conservation of total energy and momentum. n is a normal massive particle (it's the same normal particle before and after the process, e.g. the mass is the same) and t is a tachyon.

This suggests that a world containing tachyons would be unstable. None have ever been observed.

[1] With source terms, $T^{\mu\nu}_{tot} = T^{\mu\nu}_{field} + T^{\mu\nu}_{matter}$ and you can show (optional) that $\partial_\mu T^{\mu\nu}_{field} = -F^{\mu\lambda}J_\lambda$ and $\partial_\mu T^{\mu\nu}_{matter} = +F^{\nu\lambda}J_\lambda$ (this follows from $T^{\mu\nu}_{matter} = \sum_n m_n u^\mu_n u^\nu_n \delta^3(\vec{x} - \vec{x}_n(t))/\gamma_n$), so $T^{\mu\nu}_{tot}$ is conserved but the field and matter pieces aren't individually conserved if $F^{\mu\lambda}J_\lambda \neq 0$. This is because, in this case, energy and momentum can go from the fields into the matter, so only the total is conserved.