Physics 225, Homework 3, Due Monday April 25.

1. (Taken from Hartle 7.5) Consider the 2d spacetime spanned by coordinates (v, x) (the coordinate v here is like time; v is just its name, it does not denote velocity or anything like that), with the line element

$$ds^2 = -x dv^2 + 2 dv dx.$$

(a) Find the equation for the light cones ($ds^2 = 0$) for general (v, x).

(b) Draw a picture of these light cones in the (v, x) spacetime for various values of x (positive and negative).

(c) Show that a particle (non-tachyonic) can cross from positive x to negative x, but cannot cross from negative to positive x. This is similar to a black hole: there is a surface, $x = 0$, out from which you cannot get.

2. Find a coordinate transformation from

$$ds^2 = -dt^2 + 2dxdt + dy^2 + dz^2$$

into the usual flat spacetime metric $ds^2 = \eta_{\mu'\nu'} dx^{\mu'} dx^{\nu'}$.

3. (From Hartle 7.10) An observer moves on a curve $X = 2T$ for $T > 1$ in the 2d spacetime with metric

$$ds^2 = -X^2 dT^2 + dX^2 \equiv -d\tau^2.$$

Find the components of the velocity $u^\mu = \frac{d}{d\tau}(T, X)$ along this curve, for general T. Is the curve timelike?

4. Carrol 1.6. In Euclidean 3-space, let p be the point $(x, y, z) = (1, 0, -1)$ and consider the curves $x^i(\lambda) = (\lambda, (\lambda - 1)^2, -\lambda)$, and $x^i(\mu) = (\cos\mu, \sin\mu, \mu - 1)$, and $x^i(\sigma) = (\sigma^2, \sigma^3 + \sigma^2, \sigma)$.

(a) Calculate the components of the tangent vector to each of these curves at the point p in the coordinate basis $(\partial_x, \partial_y, \partial_z)$.

(b) Let $f = x^2 + y^2 - yz$. Calculate $df/d\lambda$, $df/d\mu$, and $df/d\sigma$.

5. Carroll 2.6: Consider $\mathbf{R}^3$ as a manifold with flat Euclidean metric and coordinates (x, y, z), i.e. $ds^2 = dx^2 + dy^2 + dz^2$. In spherical coordinates, the metric is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

(a) A particle moves along the curve

$$x(\lambda) = \cos\lambda, \quad y(\lambda) = \sin\lambda, \quad z(\lambda) = \lambda.$$

Express the path of this curve in the (r, θ, ϕ) coordinate system.

(b) Calculate the components of the tangent vector to the curve in both the (x, y, z) coordinate system and the (r, θ, ϕ) coordinate system.

6. (Hartle 7.14) Consider

$$ds^2 = -(1 - Ar^2)^2 dt^2 + (1 - Ar^2)^2 dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

(a) Find the proper distance along a radial line from $r = 0$ to $r = R$.

(b) Find the area of a sphere of radius R.

(c) Find the volume of a sphere of radius R.

(d) Find the 4-volume of the 4d tube given by a sphere of radius R between two $t=$ constant planes, separated by coordinate time T.

7. Consider the 2-sphere $dS^2 = d\theta^2 + \sin^2\theta d\phi^2$. Show that lines of constant longitude, (ϕ = constant) are geodesics, whereas lines of constant latitude (θ=constant) is a geodesic only for the equator, ($\theta = \pi/2$).

8. The metric for the three-sphere S^3 in coordinates $x^A = (\psi, \theta, \phi)$ can be written

$$dS^2 = d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2.)$$

Compute the Chrisoffel connection coefficients Γ^A_{BC} (where the indices A correspond to the coordiantes $x^A = (\psi, \theta, \phi)$, and likewise for B and C) by varying the integral

$$I = \tfrac{1}{2}\int g_{AB}\frac{dx^A}{dS}\frac{dx^B}{dS}dS.$$

Pick any **one** of the non-zero Γ^A_{BC} coefficients, and verify that you get the same answer for it by directly plugging into

$$\Gamma^A_{BC} = \tfrac{1}{2}g^{AD}(\partial_B g_{CD} + \partial_C g_{BD} - \partial_D g_{BC}).$$

9. (Hartle 8.9) Consider $ds^2 = -X^2 dT^2 + dX^2$. Find the shape $X(T)$ of all timelike geodesics.