5/9/11 Lecture 13 outline

• Continue with black holes! The metric of a neutral, spherically symmetric, non-rotating black hole is the good old Schwarzschild metric – the only difference is that $R_{object} \leq R_S = 2GM/c^2$, so we can consider what happens for $r \leq 2GM/c^2$. For the sun, $GM_{sun}/c^2 = 1.48km$, so the sun would be a black hole if all its mass were compressed into a radius smaller than $\sim 3km$. A planet out at R_e could continue rotating around the black hole on the same orbit as if it were a normal star – things only get bizarre on distances $\sim R_s = 2GM/c^2$. As we said last time, Consider null, radial geodesics, see they have $dt/dr = \pm(1 - \frac{2GM}{r})$, so the slope of the light cones in the (r,t) plane close up at $r = R_S = 2GM$. A light ray just outside that radius seems to never get there, but this is an illusion of the coordinate system.

Let's sit well outside R_S and drop our friend C3P0 into the horizon, and he's going to send messages back to us. Give him $\ell = 0$, so $V_{eff}(r) = \frac{1}{2}\epsilon - \epsilon GM/r$ and $\epsilon = 1$. Take him initially at $r = r_0$ at $t = t_0$ Then we have $\frac{dt}{d\tau} = e(1 - 2GM/r)^{-1}$ and $\frac{dr}{d\tau} = -\sqrt{e^2 - 2V_{eff}(r)}$. At $r = r_0$, $dt/d\tau = 1/\sqrt{1 - v_0^2} = 1$. So $e = 1 - 2GM/r_0$. The proper time to go from r_0 to r_1 is then

$$\Delta\tau_{C3P0} = \int_{r_1}^{r_1} \frac{dr}{\frac{dr}{d\tau}} = -\int_{r_0}^{r_1} \frac{dr}{\sqrt{e^2 - 1 + \frac{2GM}{r}}}.$$

The coordinate time is

$$\Delta t = \int_{r_0}^{r_1} d\tau \frac{dt}{d\tau} = -\int_{r_0}^{r_1} \frac{dr\ e}{(1 - \frac{2GM}{r})\sqrt{e^2 - 1 + \frac{2GM}{r}}}.$$

If we started him at $r \approx \infty$, then $e = 1$, and the proper time needed to go from r_1 to r_2 is

$$\Delta\tau_{C3PO} = -\int_{r_1}^{r_2} \sqrt{\frac{r}{2GM}} = \frac{2}{3\sqrt{2GM}} r^{3/2}|_{r_2}^{r_1}.$$

From any finite r_1, hit $r_2 = 2GM$ in a finite proper time, but coordinate time $\Delta t \to \infty$ for $r_2 \to R_S$. Also, the time that we see for the signals to get to us is $\Delta\tau_{US} \to \infty$. It seems to us that he never gets to the horizon.

• From the perspective of someone at $r = \infty$, the horizon is a special place. But from the perspective of someone falling in, nothing too special happens there. There are extreme tidal forces on length scales $\sim GM/c^2$, but if we imagine a supermassive black hole, this is a huge length scale and as long as the infalling observer is much smaller than that scale, they don't notice anything too special as they get too, and cross the horizon.

• One way to see that nothing special happens for a local observer at the horizon is to compute the curvature there, and show it's regular there. We'll do that shortly, after we return to discussing some math. For now, show it by using better coordinate systems there.

• Trade the original Schwarzschild time t for v, defined by $t = v - r - 2M \log |\frac{r}{2M} - 1|$, where we use $G = 1$ units. Then the metric becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

This is the same Schwarzschild geometry, but in the Eddington-Finkelstein coordinates. The geometry and physics are unchanged, only the names of the coordinates have been altered to make the physics clearer.

Let's look at the radial light cones in this coordinate system: $-(1 - \frac{2M}{r})dv^2 + 2dvdr = 0$, so we can take v =constant, i.e. $\frac{dv}{dr} = 0$, which is an ingoing light ray since increasing t means decreasing r for v =constant. Another solution is $-(1 - \frac{2M}{r})dv + 2dr = 0$, so this null curve has $\frac{dv}{dr} = 2(1 - \frac{2GM}{r})^{-1}$, or integrating, $v - 2(r + 2M \log |\frac{r}{2M} - 1|)$ =constant. This light ray is outgoing for $r > 2M$, so it's the other side of the light cone. But for $r < 2M$ it's also ingoing. Plot what's happening in the (r, v) plane. The entire light cone has been tilted, to point in toward the black hole. Uh-oh... no escape! And the horizon, $r = 2GM$, $v =$ constant, is actually a null surface.

• Let $\widetilde{t} \equiv v - r$ and plot what happens for a collapsing star in the $(r, \widetilde{t})$ plane.