6/1/11 Lecture 19 outline

• As we discussed last time, Einstein's equations

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1}$$

are 2nd order, non-linear PDEs for the metric $g_{\mu\nu}$. Today we'll briefly discuss solutions to these equations in a linearized approximation, applicable e.g. for discussing gravity waves. We'll also discuss solutions for $T^{fluid}_{\mu\nu}$, applicable for cosmology.

• Expand around flat space, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, taking $h_{\mu\nu}$ small and linearizing in it. So e.g. $g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}$. We have

$$\Gamma^{\rho}_{\mu\nu} \approx \tfrac{1}{2}\eta^{\rho\sigma}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}).$$

And we can drop the $\Gamma\Gamma$ terms in the Riemann tensor, so

$$R_{\mu\nu\rho\sigma} \approx \tfrac{1}{2}(\partial_\rho \partial_\nu h_{\mu\sigma} + \partial_\sigma \partial_\mu h_{\nu\rho} - [\mu \leftrightarrow \nu]).$$

$$R_{\mu\nu} \approx \tfrac{1}{2}(\partial_\sigma \partial_\nu h_{\mu\sigma} + \partial_\sigma \partial_\mu h^{\sigma}_{\mu} - \partial_\mu \partial_\nu h - \partial^2 h_{\mu\nu}),$$

$$R \approx \partial_\mu \partial_\nu h^{\mu\nu} - \partial^2 h.$$

Plug in to get $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R$.

Can pick names for components, $h_{00} = -2\Phi$, $h_{0i} = w_i$, and $h_{ij} = 2s_{ij} - 2\Psi\delta_{ij}$. Then $\Gamma^0_{00} = \partial_0 \Phi$, etc. The geodesic equation (taking $\lambda = \tau/m$ for massive particles)

$$\frac{dp^\mu}{d\lambda} + \Gamma^{\mu}_{\rho\sigma} p^\rho p^\sigma = 0$$

then gives, using $p^0 = dt/d\lambda = E$ and $p^i = E v^i$,

$$\frac{dp^\mu}{dt} = -\Gamma^{\mu}_{\rho\sigma} \frac{p^\rho p^\sigma}{E},$$

or in components

$$\frac{dE}{dt} = -E(\partial_0 \Phi + 2(\partial_k \Phi) v^k - (\partial_{(j} w_{k)} - \tfrac{1}{2}\partial_0 h_{jk}) v^j v^k),$$

(giving energy exchange between the particle and gravity) and

$$\frac{dp^i}{dt} = E[G^i + (\vec{v} \times H)^i - 2(\partial_0 h_{ij}) v^j - (\partial_{(j} h_{k)i} - \tfrac{1}{2}\partial_i h_{jk}) v^j v^k]$$

where $G^i \equiv -\partial_i \Phi - \partial_0 w_i$, and $H^i \equiv \epsilon^{ijk}\partial_j w_k$.

• Coordinate transformation, $\delta h_{\mu\nu} = \partial_{(\mu}\epsilon_{\nu)}$ similar to gauge transformations in E&M. Can pick convenient gauges, e.g. set $\Phi = w^i = 0$. The scalars Φ and Ψ are would-be scalars, but aren't physical. Neither is the would-be spin 1 component w_i. The only physical dof are the spin $s = 2$ quadrupole components s_{ij}. This looks like $2s+1 = 5$ components, but there's still more gauge redundancy. Actually, only 2 independent physical polarizations. Counting: $h_{\mu\nu}$ has 10 polarizations, minus 4 for $\delta x^\mu = \epsilon^\mu(x)$ symmetry, minus another 4 for the longitudinal condition, gives 2. Gauge symmetry "cuts twice," like in E&M where we have $4-1-1 = 2$, here we have $10-4-4 = 2$.

• Gravity waves in empty space. Take $T_{\mu\nu} = 0$ in Einstein's equations, and linearize them to get $\partial^2 s_{ij} = 0$. Call $h^{TT}_{\mu\nu} = 2s_{ij}$ for the i, j components and zero otherwise. Write a plane wave solution, $h^{TT}_{\mu\nu} = C_{\mu\nu}e^{ikx}$, which solves the wave equation for $k^2 = 0$: the graviton is massless. To keep it transverse (eliminate gauge dof), need $k^\mu C_{\mu\nu} = 0$. Taking $k^\mu = (\omega, 0, 0, \omega)$, find, 2 independent polarization components, $C_{11} = h_+$ and $C_{12} = h_X$. A ring of particles in the $x-y$ plane will oscillate in a $+$ shape in reaction to a gravitational wave with $h_+ \neq 0$, and $h_X = 0$. A gravitational wave with $h_X \neq 0$ and $h_+ = 0$ will cause them to oscillate in a X pattern. Can define $h_{R,L} = (h_+ \pm i h_X)/\sqrt{2}$ circular polarizations.

• Gravitational wave production. Let $I^{ij} = \int d^3x \mu(x,t) x^i x^j$ be the 2nd mass moment. The leading contribution far away, for weak sources, is

$$h_{ij} - \tfrac{1}{2} h \delta_{ij} \approx \frac{2}{r} \ddot{I}^{ij}(x,t)_{ret}.$$

Analogous to $\vec{A} \sim \dot{\vec{p}}_{ret}/r$ in E&M.

LIGO and LISA are laser interferometers, hoping to detect gravity waves.

• Now a bit of cosmology. Consider Einstein's equations with $T_{\mu\nu} = T^{fluid}_{\mu\nu} = (p+\rho)u_\mu u_\nu + p g_{\mu\nu}$. Take $p = w\rho$, where w is a constant; where the main cases are $w = 0$ for nonrelativistic matter, $w = \frac{1}{3}$ for massless matter a.k.a. radiation (recall your HW), and $w = -1$ for cosmological constant.

Consider first a cosmological constant, with $\rho = -p \equiv 3\kappa/8\pi G$. Then Einstein's equations give $R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = -3\kappa g_{\mu\nu}$ and a solution of this is $R_{\rho\sigma\mu\nu} = \kappa(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})$, which has $R_{\mu\nu} = 3\kappa g_{\mu\nu}$ and $R = 12\kappa$. If $\kappa > 0$, i.e. positive CC, then the space has positive curvature, and is called deSitter (dS) . If $\kappa < 0$, i.e. negative CC, then it's anti-de-Sitter (AdS). These are maximally symmetric spaces.

More interesting spacetimes have the RW form

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right],$$

where $a(t)$ is the scale factor and now κ is the curvature of the spatial part of the metric. If $\kappa = 0$, the spatial part is flat, while if $\kappa > 0$ it has positive curvature (like an S^3) so "closed" and if $\kappa < 0$ it has negative curvature (hyperbolic space) so "open."

Can compute the Christoffel connection and curvature of this metric, e.g. $\Gamma^0_{11} = a\dot{a}/(1-\kappa r^2)$, $\Gamma^3_{03} = \dot{a}/a$, $\Gamma^1_{11} = \kappa r/(1-\kappa r^2)$, etc and $R_{00} = -3\ddot{a}/a$ etc., and $R = 6(\ddot{a}/a + (\dot{a}/a)^2 + \kappa/a^2)$.

• Now consider $T_{\mu\nu} = T^{fluid}_{\mu\nu} = (p+\rho)u_\mu u_\nu + p g_{\mu\nu}$ with this spacetime metric. Note that $0 = \nabla^\mu T_{\mu\nu}$, for $\nu = 0$ gives $0 = -\partial_0\rho - 3\frac{\dot{a}}{a}(\rho + p)$. Setting $p = w\rho$, conservation of energy becomes

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a},$$

so $\rho_M \sim a^{-3(1+w)}$. For matter, $w = 0$, and $\rho \sim a^{-3}$, i.e. the matter energy density dilutes as the space grows, proportional to the scale factor a, fitting with fixed amount of stuff. For radiation, $w = \frac{1}{3}$, get $\rho_R \sim a^{-4}$. This also makes sense: as the space grows $\sim a$, there is an extra energy density suppression factor of a compared with the matter case, because the wavelength of the radiation is being redshifted $\sim a$. Finally, for CC, get $\rho \sim a^0$, the energy density of vacuum is independent of the scale factor.

Einstein's equation for the 00 component gives

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

and for the spatial part ij gives another equation that leads to the other Friedman equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2}.$$

The Hubble parameter is defined by $H = \dot{a}/a$ and currently $H_0 \approx 70 km/sec/MPc$, where $Mpc = 3 \times 10^{24} cm$. In particle physics units, $H_0 \approx 10^{-33} eV$.

Define $\rho_{crit} = 3H^2/8\pi G$ and $\Omega = 8\pi G\rho/3H^3 = \rho/\rho_{crit}$, and then $\Omega - 1 = \kappa/H^2 a$.

Write $\rho_i = \rho_{i0} a^{-n_i}$, where $w_i = \frac{1}{3}n_i - 1$, and then the Friedman equation gives $H^2 = \frac{8\pi G}{3}\sum \rho_i$, where curvature is included as $\rho_c = -3\kappa/8\pi G a^2$, with $w_i = -\frac{1}{3}$. The Friedman equation gives $a \sim t^{2/n}$. Matter dominated: $a = (t/t_0)^{2/3}$; radiation dominated, $a = (t/t_0)^{1/2}$; vacuum dominated: $a = e^{H(t-t_0)}$, where $H = 8\pi\rho/3 = \Lambda/3$.

If $\Omega_M + \Omega_\Lambda = 1$, then $\kappa = 0$, and the universe is flat. If $\Omega_M + \Omega_\Lambda > 0$ there is positive spatial curvature, $\kappa > 0$. Get

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_M - 2\rho_\Lambda).$$

Einstein tried to get a static universe, so he imagined (his self-described "greatest blunder") that $\rho_M = 2\rho_\Lambda$. If $\rho_M - 2\rho_\Lambda > 0$, the universe decelerates and eventually re-collapses. Observation fits with $2\rho\Lambda - \rho_M > 0$, so the universe has accelerated expansion. Over time $\rho_M \to 0$, and ρ_Λ =constant, so it accelerates more in the future. Observation fits with, presently, $\Omega = 1$: $\Omega_{DE} \approx 75\%$, $\Omega_{DM} \approx 21\%$, $\Omega_{NM} \approx 4\%$.

• Consider a scalar field in the RW metric. Its equations of motion become, dropping the spatial derivative terms,

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

the second term is "Hubble friction" from the Christoffel connection (can use $\nabla_\mu \nabla^\mu \phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu \phi)$ with $\sqrt{-g} = a^3$.) . The Friedman equation now gives

$$H^2 = \frac{1}{3m_P^2}(\tfrac{1}{2}\dot{\phi}^2 + V(\phi).).$$

Inflation is built from such scalars with sufficiently gradual potentials so that the scalar field rolls down very slowly, and the potential looks approximately like a CC. The slow rolling then leads to $H \approx$ constant, an exponential expansion of space.

• Take 225b next quarter for more!