

4/4/11 Lecture 3 outline

- Summarize from last time. Examples of 4-scalars: ds^2 , $a_\mu b^\mu$, $d\tau = \sqrt{-ds^2}$, mass, the action, Q_{encl} . Examples of 4-vectors: dx^μ , ∂^μ (remember $\partial_\mu = (+\frac{\partial}{\partial t}, \nabla)$), $p^\mu = (E, \vec{p})$, $k^\mu = (\omega, \vec{k})$, $u^\mu = \frac{dx^\mu}{d\tau}$, $f^\mu = \frac{dp^\mu}{d\tau} = (\gamma P_{ower}, \gamma \vec{F})$, $J^\mu = (\rho, \vec{j})$, $J^\mu = \rho \frac{dx^\mu}{dx^0}$, $A^\mu = (\phi, \vec{A})$. Examples of 4 tensors: energy-momentum $T^{\mu\nu}$ (symmetric), $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

Equations of motion must transform covariantly, so all observers agree if they're satisfied. This follows from the EL equations, since S is Lorentz invariant

- Last time, recall that for a time-like path $d\tau = \sqrt{-ds^2}$ and in another frame have $dt = \gamma d\tau$. For timelike paths have $u^\mu = dx^\mu/d\tau$, with $u^\mu = (\gamma, \gamma \vec{v})$, so $u^\mu u_\mu = -1$.

- For lightlike paths, write $u^\mu(\lambda) = dx^\mu/d\lambda$, with $u^\mu u_\mu = 0$. Here λ just parameterizes the path and there isn't any physical meaning to it; indeed, we can reparameterize $\lambda \rightarrow \lambda'(\lambda)$ and nothing changes.

- As discussed last time, $P_{encl}^\mu = \int dV T^{\mu 0}$. For a perfect fluid, $T^{\mu\nu} = diag(\rho, p, p, p)$, in the rest frame. So $T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p\eta^{\mu\nu}$. Illustrates a nice technique: find tensor expression from starting in the rest frame. (Could also get it directly, by Lorentz transforming from the rest frame to one moving with velocity u^μ through the fluid. But the above applies even in a non-inertial frame, e.g. if $u^\mu(\tau)$ is the 4-velocity of an accelerated observer.)

Example: $T_{CC}^{\mu\nu} = -\Lambda\eta^{\mu\nu}$, has $\rho_{CC} = -p_{CC}$.

Back to E&M: $\frac{dp^\mu}{d\tau} = qF^{\mu\nu}u_\nu$, where $F^{0i} = E^i$ and $F^{ij} = \epsilon^{ijk}B_k$. Here $F^{\mu\nu} = -F^{\nu\mu}$. Under Lorentz transformations, $F^{\mu'\nu'} = \Lambda_{\rho'}^{\mu'}\Lambda_{\sigma'}^{\nu'}F^{\rho\sigma}$. Maxwell's equations are $\partial_\mu F^{\nu\mu} = J^\nu$ (which implies $\partial_\nu J^\nu = 0$), and $\partial_\mu F_{\nu\lambda} + cyclic = 0$. Solve the second via $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with $A^\mu = (\phi, \vec{A})$. Note gauge invariance $A^\mu \rightarrow A^\mu + \partial^\mu f$. In Coulomb gauge take $\partial_\mu A^\mu = 0$ and then get $\partial^2 A^\nu = -J^\nu$. Plane wave solutions like $A^\mu = \epsilon^\mu(k)e^{ik \cdot x}$.

For a massive charged particle, $S = -m \int d\tau + q \int A_\mu dx^\mu$. Gives $\vec{p} = \partial L / \partial \vec{v} = \gamma m \vec{v} + q \vec{A}$, and $H = \vec{p} \cdot \vec{v} - L = \gamma m + q\phi = \sqrt{m^2 + (\vec{p} - q\vec{A})^2} + q\phi$.

- Classical field theory, e.g. for a scalar field: $S = \int d^4x \mathcal{L}(\Phi, \partial_\mu \Phi)$, with EL equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} - \frac{\partial \mathcal{L}}{\partial \Phi} = 0.$$

Example, $\mathcal{L} = -\frac{1}{2}\partial^\mu \Phi \partial_\mu \Phi - V(\Phi)$, get EL equations $(\partial_t^2 - \nabla^2)\Phi + \frac{dV}{d\Phi} = 0$.

In E&M, we have instead a classical field theory for $A^\mu(x)$, $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu$. Varying w.r.t. A_μ , the EL equations give the Maxwell equations $\partial_\mu F^{\nu\mu} = J^\nu$.

The energy-momentum tensor is the conserved Noether current related to space-time translation invariance. Get

$$T_{E\&M}^{\mu\nu} = F^{\mu\lambda} F_{\lambda}^{\nu} - \frac{1}{4} \eta^{\mu\nu} F^{\lambda\sigma} F_{\lambda\sigma}.$$

- Using u^{μ} to pick out the $a^{\mu=0}$ component is a useful general technique. Example of Hans Solo's view of frequency of some star (at say $x_1 = -\infty$). In lab frame, it's ω_* . Hans sees $\omega_{obs} = -k \cdot u_{obs}$, which can be evaluated in the lab, $\omega_{obs} = \omega_*(u^0 - u^1)$, where u^{μ} is the 4-velocity of the Millennium Falcon, as measured in the lab frame. For example, suppose $t(\tau)_{obs} = a^{-1} \sinh a\tau$ and $x(\tau)_{obs} = a^{-1} \cosh a\tau$, so $u_{obs}^0 = \cosh a\tau$ and $u_{obs}^1 = \sinh a\tau$, then we get $\omega(\tau) = \omega_* e^{-a\tau}$. At later and later times, the light is more and more red-shifted.

- Equivalence principle

WEP: $m_i = m_g$. LHS: enters in $\vec{F} = m_i \vec{a}$, and RHS enters in $\vec{F}_g = -m_g \nabla \Phi$, so $\vec{a} = -\nabla \Phi$. Can't distinguish between gravity and acceleration. Motion of freely falling particles locally same in gravity field vs a uniformly accelerating frame. Eotvos experiment.

EEP: In small regions laws of physics reduce to special relativity, freely falling small observers can't detect gravity by any local experiment.

E.g. binding energy of hydrogen atom contributes s.t. $m_i = m_g$ is preserved.

SEP: **All** laws of physics are such that gravity can't be detected by local, free falling observer's experiments. E.g. gravitational binding energy contributes equally to inertial and gravitational mass. Is it true? Maybe yes, maybe no - some theories might allow for small deviations. Tom Murphy is checking this, by precise lunar ranging!