4/20/11 Lecture 8 outline

• From now on, if I say that some quantity "transforms properly," it means transforms covariantly under general coordinate transformations $x^{\mu'}(x)$, i.e. all indices transform with the appropriate factor of "$\frac{\partial x}{\partial x}$". GR is based on the symmetry principle of general coordinate invariance, and it posits that all physical quantities transform properly.

Examples of properly transforming vectors (components) $u^\mu = \frac{dx^\mu}{d\lambda}$, J^μ, A^μ, p^μ, k^μ etc. Again, we can always raise / lower indices using the metric $g_{\mu\nu}$ or its inverse $g^{\mu\nu}$.

We saw that, if A_μ is a properly transforming vector, then $\partial_\nu A_\mu$ is not, because of a bad term. The bad term cancels in $\partial_\nu A_\mu - \partial_\mu A_\nu$. We'll see shortly how to correct for the bad term. Basically, we'll see that we need to replace ∂_μ with an improved quantity called $\nabla_\mu = \partial_\mu + \ldots$, where the $\ldots$ is something that'll depend on what it's acting on.

• Equivalence principle → a particle in a general spacetime metric moves to extremize

$$\Delta\tau = \int d\tau = \int d\lambda \sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}/c^2}. \tag{1}$$

This is the definition of a geodesic. It applies whether the particle is massive or massless. Note that for massive particles we can replace $\lambda \to \tau$, while for massless ones we keep it as an arbitrary parameter. In any case, note reparameterization symmetry $\lambda \to \lambda'(\lambda)$. The geodesic equation can be derived from the E.L. equations:

$$\delta\Delta\tau = -\tfrac{1}{2}\int (-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda})^{-1/2}\delta(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}),$$

so we need $\delta(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}) = 0$. (Note that we'd get these same EOM if we extremized instead $\int d\lambda\ g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$.) This gives

$$0 = \delta x^\rho \left[\partial_\rho g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - 2\frac{d}{d\lambda}(g_{\rho\nu}\frac{dx^\nu}{d\lambda})\right].$$

Using $\frac{dg_{\rho\nu}}{d\lambda} = \partial_\sigma g_{\rho\nu}\frac{dx^\sigma}{d\lambda}$ and noting that we need to symmetrize $\frac{dx^\sigma}{d\lambda}\frac{dx^\nu}{d\lambda}$ in σ and ν,

$$\delta ds^2/d\lambda^2 = -2g_{\rho\nu}\delta x^\rho \left[\frac{d^2x^\nu}{d\lambda^2} + \Gamma^\nu_{\mu\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\sigma}{d\lambda}\right],$$

where we here meet the Christoffel symbol (which we'll later use to fix $\partial_\mu A_\nu$ not being a vector)

$$\Gamma^\nu_{\mu\sigma} = \tfrac{1}{2}g^{\rho\nu}(\partial_\mu g_{\rho\sigma} + \partial_\sigma g_{\rho\mu} - \partial_\rho g_{\mu\sigma})$$

So the geodesic equation is

$$\frac{d^2x^\nu}{d\lambda^2} + \Gamma^\nu_{\mu\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0.$$

Writing $u^\mu \equiv \frac{dx^\mu}{d\lambda}$, the geodesic equation is $\frac{d}{d\lambda}u^\nu = -\Gamma^\nu_{\mu\sigma}u^\mu u^\sigma$.

In free-falling coordinates, we can make locally at any point p $g_{\widetilde{\mu\nu}}|_p = \eta_{\widetilde{\mu\nu}}$ and $\partial_{\widehat{\sigma}} g_{\widetilde{\mu\nu}}|_p = 0$, so we can make $\Gamma^{\widehat{\nu}}_{\widetilde{\mu\sigma}}|_p = 0$, and the geodesic equation looks like free-fall in special relativity. As we saw in our example of the metric at the North pole of the sphere, quadratic terms in the metric away from p don't vanish, so generally e.g. $\partial_{\widehat{\kappa}}\Gamma^{\widehat{\nu}}_{\widetilde{\mu\sigma}}|_p \neq 0$. We'll see a little later that these terms are indeed sensitive to local curvature.

• A preview of something we'll discuss more later: we mentioned that $\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}$ gives a geometric vector when acting on invariant functions. However, it has to be modified when acting on vectors, corresponding to the fact that geometric vectors need to be parallel transported around the geometry to behave properly. We'll introduce $\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu$, where $\nabla_\mu = \partial_\mu + \ldots$, where the $\ldots$ depends what it's acting on, they aren't there for scalars but are needed for vectors, dual vectors, or general tensors. In this notation, the geodesic equation is

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0.$$

• If the particle is not free falling, i.e. if it is under the influence of another force, we just add the appropriate force term, to the RHS of the geodesic equation, e.g. a charged particle in external electric and magnetic fields have

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\nu_{\mu\sigma}\frac{dx^\mu}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{q}{m}g_{\nu\kappa}\frac{dx^\nu}{d\tau}F^{\kappa\mu}.$$

• Examples of geodesic equation (Hartle ch 8).

Plane in polar coodinates, $dS^2 = dr^2 + r^2 d\phi^2$. Geodesics extremize $\Delta S = \int dS = \int d\sigma L$, with $L = \frac{dS}{d\sigma} = \sqrt{\dot{r}^2 + r^2\dot{\phi}^2}$, where here we use $\dot{=}\frac{d}{d\sigma}$. The EL equations are the same if we forget about the $\sqrt{}$. We can also multiply the EL equations by $\frac{d\sigma}{dS}$, which essentially just replaces $\sigma \to S$ as the curve's parameter. The EL equations then give $r = r(S)$ and $\phi = \phi(S)$ with

$$\frac{d^2r}{dS^2} = r(\frac{d\phi}{dS})^2, \qquad \frac{d}{dS}\left(r^2\frac{d\phi}{dS}\right) = 0,$$

which are geodesic equations with $\Gamma^r_{\phi\phi} = -r$ and $\Gamma^\phi_{r\phi} = 1/r$. This illustrates how the Christoffel connection can be obtained either from its definition directly in terms of the metric or, sometimes more conveniently, directly from the geodesic equation. Working directly in terms of the metric, with $g_{rr}=1$ and $g_{\phi\phi}=r^2$, easily verify e.g. $g_{rr}\Gamma^r_{\phi\phi}=-r$.

• When we try to solve the geodesic equations, it's useful to use integrals of the motion. While it's perhaps not yet obvious (we'll explain it more soon), one integral of the motion is

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}=\text{constant}.$$

For massive particles we can take $\lambda=\tau$ and the constant is -1, $u_\mu u^\mu=-1$. For photons, the constant is 0, showing that photons move on the light cone.

When the metric has a spatial symmetry isometry, then there is another integral of the motion, $\xi\cdot u=0$ where ξ is any Killing vector isometry of the space.

In our example above, $u\cdot u=-1$ is replaced with $\vec{u}\cdot\vec{u}=1$, i.e. $r'^2+r^2\phi'^2=1$, where $'=\frac{d}{dS}$. There is an isometry of rotating ϕ, so $\xi^r=0$, $\xi^\phi=1$, and the conserved quantity is $\ell\equiv\vec{\xi}\cdot\vec{u}=g_{AB}\xi^Au^B=r^2\phi'$. So the first equation becomes $r'=\sqrt{1-\ell^2/r^2}$ and then the second gives $\phi'/r'=\frac{d\phi}{dr}=\ell r^{-2}(1-\ell^2/r^2)^{-1/2}$, which integrates to $\phi=\phi_*+\cos^{-1}(\ell/r)$. At the end, this yields the straight lines expected from rectangular coordinates, the hard way.

• Example, FRW metric $ds^2=-dt^2+a^2(t)(dx^2+dy^2+dz^2)$. Consider extremizing $\int d\tau(-(\frac{dt}{d\tau})^2+a^2(t)\frac{dx^i}{d\tau}\frac{dx^i}{d\tau})$, (it gives the same EOM as the $\sqrt{}$). Vary δt and $\delta\vec{x}$ to get the four EOM equations:

$$\frac{d^2x^0}{d\tau^2}+a\dot{a}\frac{dx^i}{d\tau}\frac{dx_i}{d\tau}=0.$$

$$\frac{d^2x^i}{d\tau^2}+\frac{2\dot{a}}{a}\frac{dx^i}{d\tau}\frac{dx^0}{d\tau}=0.$$

So we see that $\Gamma^0_{ij}=a\dot{a}\delta_{ij}$ and $\Gamma^i_{0j}=\frac{\dot{a}}{a}\delta_{ij}$ are the non-zero terms (which also follows from plugging the metric into the formula above).

Consider e.g. the null paths of photons moving along the x axis, $(t(\lambda),x(\lambda),0,0)$ with $\frac{dx}{d\lambda}=\frac{1}{a}\frac{dt}{d\lambda}$. Combine this with the geodesic equation for $x^0=t$ to get

$$\frac{d^2t}{d\lambda}^2+\frac{\dot{a}}{a}(\frac{dt}{d\lambda})^2=0.$$

The solution is $\frac{dt}{d\lambda}=\omega_0/a$. This is good. The 4-velocity of some observer is u^μ with $u_\mu u^\mu=-1$ and in their rest frame $u^0=1/\sqrt{-g_{00}}$ and the energy of a photon is $E=-p_\mu u^\mu=-g_{00}u^0\frac{dx^0}{d\lambda}$. So the energy measured by this comoving observer at fixed spatial coordinates is $-\sqrt{g_{00}}\frac{dx^0}{d\lambda}=\omega_0/a\sim 1/a$. This is the cosmological redshift.

• Next topic, study $ds^2=-(1-2GM/rc^2)dt^2+(1-2GM/rc^2)^{-1}dr^2+r^2d\Omega^2$.