5/16/12 Lecture outline

$\star$ Reading: Zwiebach chapter 10.

• Last time, consider classical scalar field theory, with $S = \int d^Dx(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2)$. The EOM is the Klein-Gordon equation

$$(\partial^2 - m^2)\phi = 0, \qquad \partial^2 \equiv -\frac{\partial^2}{\partial t^2} + \nabla^2$$

The Hamiltonian is $H = \int d^{D-1}x(\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2)$, where $\Pi = \partial\mathcal{L}/\partial(\partial_0\phi) = \partial_0\phi$. Take e.g. $D = 1$ and get SHO with $q \to \phi$ and $m \to 1$ and $\omega \to m$.

Classical plane wave solutions: $\phi(t, \vec{x}) = ae^{-iEt+i\vec{p}\cdot\vec{x}} + c.c.$, where $E = E_p = \sqrt{\vec{p}^2 + m^2}$, and the $+c.c.$ is to make ϕ real. Letting $\phi(x) = \int \frac{d^Dp}{(2\pi)^D}e^{ip\cdot x}\phi(p)$, the reality condition is $\phi(p)^* = \phi(-p)$ and the EOM is $(p^2 + m^2)\phi(p) = 0$.

• Now consider light cone gauge coordinates. Replace $\partial^2 \to -2\partial_+\partial_- + \partial_I\partial_I$ and Fourier transform in all coordinates except x_+:

$$\phi(x^+, x^-, \vec{x}_T) = \int \frac{dp^+}{2\pi}\int \frac{d^{D-2}\vec{p}_T}{(2\pi)^{D-2}}e^{-ix^-p^+ + i\vec{x}_T\cdot\vec{p}_T}\phi(x^+, p^+, \vec{p}_T).$$

Then the EOM becomes

$$(i\frac{\partial}{\partial x^+} - \frac{1}{2p^+}(p^Ip^I + m^2))\phi(x^+, p^+, \vec{p}_T) = 0.$$

Looks like the non-relativistic Schrodinger equation, with x_+ playing the role of time and p^+ playing the role of mass, even though it is secretly relativistic.

• Let's quantize! Replace ϕ with an operator. Consider

$$\phi(t, \vec{x}) = \frac{1}{\sqrt{V}}\sum_{\vec{p}}\frac{1}{\sqrt{2E_p}}(a_{\vec{p}}(t)e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}(t)e^{-ip\cdot x}).$$

If we're in a spatial box, then $p_iL_i = 2\pi n_i$. Compute the energy to find

$$H = \sum_{\vec{p}>0}(\frac{1}{2E_p}\dot{a}^\dagger\dot{a}(t) + \tfrac{1}{2}E_pa^\dagger a) = \sum_{\vec{p}}E_pa^\dagger_{\vec{p}}a_{\vec{p}}.$$

where the EOM were used in the last step: $a_{\vec{p}}(t) = a_{\vec{p}}e^{-iE_pt} + a^\dagger_{-\vec{p}}e^{iE_pt}$. Also,

$$\vec{P} = \sum_{\vec{p}}\vec{p}a^\dagger_{\vec{p}}a_{\vec{p}}.$$

As expected, H and $\vec{P}$ are independent of t. We quantize this as a (complex) SHO for each value of $\vec{p}$:

$$[a_p, a_k^\dagger] = \delta_{p,k}, \qquad [a_p, a_k] = [a_p^\dagger, a_k^\dagger] = 0.$$

and interpret the above H and $\vec{P}$ has saying that $a_{\pm\vec{p}}^\dagger$ is a creation operator, creating a state with energy $E_p = \sqrt{\vec{p}^2 + m^2}$ and spatial momentum $\vec{p}$ from the vacuum $|\Omega\rangle$. (Note that we dropped the $2 \cdot \frac{1}{2} E_p$ groundstate energy contribution, for no good reason. This is the road to the unresolved cosmological constant problem, so we won't go there.)

• Now consider the Maxwell field A^μ and quantize $\to$ photons. Maxwell's equations in vacuum are $\partial_\nu F^{\mu\nu} = 0$, which with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ give $\partial^2 A^\mu - \partial^\mu(\partial \cdot A) = 0$. In momentum space, $p^2 A^\mu - p^\mu(p \cdot A) = 0$. The gauge transformation $A_\mu \to A_\mu + \partial_\mu f$ becomes $\delta A_\mu(p) = i p_\mu f(p)$. In light cone gauge, we use this freedom to set $A^+(p) = 0$. Taking $\mu = +$ in the EOM then implies $(p \cdot A) = 0$ (which implies that $A^- = (p^I A^I)/p^+$) and it then also follows that $p^2 A^\mu(p) = 0$. So we see that there are $D-2$ possible photon polarizations, with the 1-photon states given by $\sum_{I=2}^{D-1} \xi_I a_{p^+, \vec{p}_T}^{I\dagger} |\Omega\rangle$.

• Gravitational light cone gauge conditions: $h^{++} = h^{+-} = h^{+I} = 0$. Other light cone components are constrained. So physical d.o.f. are specified by a traceless symmetric matrix h^{IJ} in the $D-2$ transverse directions. So $\frac{1}{2}(D-2)(D-1) - 1 = \frac{1}{2}D(D-3)$ d.o.f..