4/11/12 Lecture outline

⋆ Reading: Zwiebach chapters 2 and 3.

• Last time: e.g. $p^{\pm} = \frac{1}{\sqrt{2}}(p^0 \pm p^1) = -p_{\mp}$. So $i\hbar\partial_{x^+} \to -p_+ = E_{lc}/c$, i.e. $p^- = E_{lc}/c$.

Extra (spacelike) dimensions, e.g. 2 extra dimensions: $-ds^2 = -c^2dt^2 + \sum_{i=1}^{5}(dx^i)^2$. Consider one extra space dimension, taken to be a circle, $x \sim x + 2\pi R$. Now consider $(x, y) \sim (x+2\pi R, y) \sim (x, y+2\pi R)$; gives a torus. Orbifold, e.g. $z \sim e^{i\pi i/N}z$, gives a cone (singular at fixed point).

Recall QM: $[x^i, p_j] = i\hbar\delta^{i_j}$. Particle in square well box of size a: $E = (n\pi/a)^2/2m$. Now particle in periodic box, $x_4 \sim x_4 + 2\pi R$. The other directions, x^μ, are given by some standard Hamiltonian, e.g. the hydrogen atom, which we'll call H_{4d}. So $H_{5d} = H_{4d} + \widehat{p}_4^2/2m$, with $\widehat{p}_4 = -i\hbar\partial_{x_4}$ in position space. The 4d energy eigenstates are then given by separation of variables to be $\psi_{E_{5d}}(\vec{x}, x_4) = \psi_{E_{4d}}(\vec{x})\frac{1}{\sqrt{2\pi R}}e^{i\ell x_4/R}$, with ℓ an integer, and $\psi_{E_{4d}}$ is an energy eigenstate of the 4d problem. So $E_{5d} = E_{4d} + \ell^2/2mR^2$. For R small, the low energy states are simply those with $\ell = 0$, and the extra dimension is unseen.

• Use units where Maxwell's equations are $\nabla \times \vec{E} = -\frac{1}{c}\partial_t\vec{B}$, $\nabla \cdot \vec{B} = 0$, $\nabla \cdot \vec{E} = \rho$, $\nabla \times \vec{B} = \frac{1}{c}\vec{j} + \frac{1}{c}\partial_t\vec{E}$. The first two equations can be solved by introducing the scalar and vector potential: $\vec{B} = \nabla \times \vec{A}$, $\vec{E} = -\frac{1}{c}\partial_t\vec{A} - \nabla\phi$. There is a redundancy here, called invariance under gauge transformation, because the physical quantities $\vec{E}$ and $\vec{B}$ are invariant under

$$\phi \to \phi - \frac{1}{c}\frac{\partial f}{\partial t}, \qquad \vec{A} \to \vec{A} + \nabla f, \tag{1}$$

for an arbitrary function $f(t, \vec{x})$. This initially dull sounding invariance takes a fundamental role in modern high energy physics: such local (because f can vary locally over space-time) gauge symmetries are in direct correspondence with forces!

• Maxwell's equations in relativistic form. Like last time, $x^\mu = (ct, \vec{x})$ and also use $\partial_\mu = (c\partial_t, \nabla)$ (and thus $\partial^\mu = (-c\partial_t, \nabla)$). $\vec{E}$ and $\vec{B}$ combine into an antisymmetric, 2-index, 4-tensor $F_{\mu\nu} = -F_{\nu\mu}$, via $F_{0i} = -E_i$ and $F_{ij} = \epsilon_{ijk}B^k$, i.e.

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}.$$

As usual, we can raise and lower indices with $\eta_{\mu\nu}$, e.g. $F^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\sigma}F_{\lambda\sigma}$ and with the book's sign convention this gives a minus sign each time a time component is raised or lowered. So $F^{0i} = -F_{0i}$ and $F^{ij} = F_{ij}$, where i and j refer to the spatial components, i.e. the matrix $F^{\mu\nu}$ is similar to that above, but with $\vec{E} \to -\vec{E}$.

Under Lorentz transformations, $x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu}$, the electric and magnetic fields transform as $F^{\mu'\nu'} = \Lambda^{\mu'}{}_{\sigma}\Lambda^{\nu'}{}_{\rho}F^{\sigma\rho}$. Sources combine into a 4-vector as $j^{\mu} = (c\rho, \vec{j})$, and charge conservation is the Lorentz-invariant equation $\partial_{\mu} j^{\mu} = 0$. Maxwell's equations in relativistic form are $\partial_{[\mu} F_{\rho\sigma]} = 0$, and $\partial_{\lambda} F^{\mu\lambda} = \frac{1}{c} j^{\mu}$ (this convention, with indices not next to each other contracted, is peculiar to the $(- + ++)$ choice of $\eta_{\mu\nu}$), which exhibits that they transform covariantly under Lorentz transformations. The scalar and vector potential combine to the 4-vector $A^{\mu} = (\phi, \vec{A})$ and the first two Maxwell equations are solved via $F^{\mu\nu} = \partial^{[\mu} A^{\nu]}$. The gauge invariance is $A^{\mu} \to A^{\mu} + \partial^{\mu} f$. We can e.g. choose Lorentz gauge, where $\partial_{\mu} A^{\mu} = 0$. Physics is independent of choice of gauge, but some are sometimes more convenient than others along the way, depending on what's being done. In Lorentz gauge, the remaining Maxwell equations are $\partial_{\mu}\partial^{\mu} A^{\nu} = -\frac{1}{c} j^{\nu}$ (still some gauge freedom). In empty space we set $j^{\mu} = 0$ and the plane wave solutions are $A^{\mu} = \epsilon^{\mu}(p) e^{ip\cdot x}$, where $p^2 = 0$ (massless) and $p \cdot \epsilon = 0$. Can still shift $\epsilon^{\mu} \to \epsilon^{\mu} + \alpha p^{\mu}$, so 2 independent photon polarizations ϵ^{μ}.

• The action of a relativistic point particle in the presence of electric and magnetic fields is

$$S = \int (-mc ds + \frac{q}{c} A_{\mu} dx^{\mu}), \tag{2}$$

which is manifestly relativistically invariant. Note also that, under a gauge transformation, we have $S \to S + \frac{qf}{c}$, which does not affect the equations of motion (just as changing the Lagrangian by a total time derivative does not).

The lagrangian is thus $L = -mc\sqrt{1 - \vec{v}^2/c^2} + \frac{q}{c}\vec{v} \cdot \vec{A} - q\phi$. The momentum conjugate to $\vec{r}$ is $\vec{P} = \partial L/\partial \vec{v} = m\vec{v}/\sqrt{1 - \vec{v}^2/c^2} + \frac{q}{c}\vec{A}$. The Hamiltonian is $H = \vec{v} \cdot \vec{P} - L = \sqrt{m^2c^4 + c^2(\vec{P} - \frac{q}{c}\vec{A})^2} + q\phi$.