4/24/13 Lecture outline

$\star$ Reading: Zwiebach chapters 4, 5.

• Nonrelativistic strings. $[T_0]=[F]=[E]/L=[\mu_0][v^2]$. Indeed, considering $F=ma$ for an element dx of the string yields the string wave equation $\frac{\partial^2 y}{\partial x^2}-\frac{1}{v_0^2}\frac{\partial^2 y}{\partial t^2}=0$, with $v_0=\sqrt{T_0/\mu_0}$. Endpoints at $x=0$ and $x=a$. Can choose Dirichlet or Neumann BCs at these points. With Dirichlet at each end, $y_n(x)=A_n\sin(n\pi x/a)$ and the general solution is $y(x,t)=\sum_n y_n(x)\cos\omega_n t$, where $\omega_n=v_0 n\pi/a$ (and the A_n are determined from the initial conditions, by Fourier transform).

The nonrelativistic string action is $S=\int dtL$ where L is the kinetic energy minus potential energy, which gives

$$S=\int dt\int dx\left(\tfrac{1}{2}\mu_0(\frac{\partial y}{\partial t})^2-\tfrac{1}{2}T_0(\frac{\partial y}{\partial x})^2\right),$$

which is a particular case of the more general action $S=\int dtdx\mathcal{L}(\frac{\partial y}{\partial t},\frac{\partial y}{\partial x})$. We can then define the momentum density and corresponding spatial quantity

$$\mathcal{P}^t=\frac{\partial\mathcal{L}}{\partial\dot{y}},\qquad \mathcal{P}^x=\frac{\partial\mathcal{L}}{\partial y'}.$$

The variation of the action is

$$\delta S=\int dtdx[\mathcal{P}^t\delta\dot{y}+\mathcal{P}^x\delta y']=-\int dtdx[\frac{\partial\mathcal{P}^t}{\partial t}+\frac{\partial\mathcal{P}^x}{\partial x}]\delta y+\text{bndy terms}$$

and the action is made stationary, $\delta S=0$, if the boundary terms vanish and if

$$\frac{\partial\mathcal{P}^t}{\partial t}+\frac{\partial\mathcal{P}^x}{\partial x}=0,$$

which when applied to the above particular choice of action gives the usual wave equation. The boundary terms must also be set to zero, and they involve $\mathcal{P}^t\delta y$ at the time endpoints and $\mathcal{P}^x\delta y$ at the space endpoints. Neumann BCs is to set $\mathcal{P}^x=0$ at the spatial endpoints (for all t), and Dirichlet BCs is to set $\delta y=0$ (and thus $\mathcal{P}^t=0$) at the spatial endpoints.

• The action for a relativistic point particle of mass m is $S=-mc\int ds=-mc^2\int dt\sqrt{1-v^2/c^2}$. This gives $\vec{p}=\partial_{\vec{v}}=\gamma m\vec{v}$ and $H=\vec{p}\cdot\vec{v}-L=\gamma mc^2$, both of which are constants of the motion (thanks to the time and spatial translation invariance).

• Reparametrization invariance: write $x_\mu(\tau)$, and can change worldline parameter τ to an arbitrary new parameterization $\tau'(\tau)$, and the action is invariant. To see this use $S=-mc\int\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$ and change $\frac{dx^\mu}{d\tau}=\frac{dx^\mu}{d\tau'}\frac{d\tau'}{d\tau}$ and note that $S\to S$. The Euler Lagrange equations of motion are $\frac{dp_\mu}{d\tau}=0$.