4/10/17 Lecture 3 outline

• Last time: $S = -mc^2 \int d\tau$ and L for relativistic particle. Relativistic charged particle $S \supset -(q/c)\int A_\mu dx^\mu$ and term in $L \supset -q\phi + (q/c)\vec{v}\cdot\vec{A}$. Canonical momentum $\vec{p} = \partial L/\partial\vec{v} = \gamma m\vec{v} - q\vec{A}/c$ and $H = \vec{p}\cdot\vec{v} - L = \gamma mc^2 - q\phi$ (magnetic fields do no work). But H should be expressed in terms of $\vec{p}$.

• QM: replace $\widehat{p}^\mu \to i\hbar\partial^\mu$ in position space. Consider the S.E.

• Punchline: in non-rel limit get $i\hbar D^0\psi = (\hbar^2/2m)\vec{D}^2\psi$, where $D^\mu \equiv (D^0, \vec{D}) = \partial^\mu - (q/i\hbar c)A^\mu$.

Get $p^\mu \to p^\mu - (q/c)A^\mu$ when a charged particle is in an $\vec{E}$ and $\vec{B}$ field. For QM, $p^\mu \to i\hbar\partial^\mu$ in position space, so get e.g. $i\hbar D^0\psi = (-\hbar^2/2m)\vec{D}^2\psi$, where (punchline) $D^\mu \equiv (D^0, \vec{D}) = \partial^\mu - (q/i\hbar c)A^\mu$ is the covariant derivative.

• $S = \ldots - q/c\int A_\mu dx^\mu$ and gauge invariance.

• Gauge transformations $A_\mu \to A_\mu + \partial_\mu f(x)$, and local $U(1)$ phase rotation of $\psi \to e^{-iq/\hbar c}\psi$, with $D_\mu\psi \to e^{-iq/\hbar c}D_\mu\psi$.

• Hamiltonian picture: $i\hbar\partial_t U(t,t_0) = HU(t,t_0)$, $i\hbar|\psi(t)\rangle_S = H|\psi(t)\rangle_S$, and $\mathcal{O}^H = U^\dagger \mathcal{O} U$ has

$$\frac{d}{dt}\mathcal{O}^H = \frac{1}{i\hbar}[\mathcal{O}^H, H] + \frac{\partial}{\partial t}\mathcal{O}^H.$$

Consider e.g. the SHO, $H = p^2/2m + \frac{1}{2}m^2\omega^2x^2$. The equation $H|n\rangle = E_n|n\rangle$ in position space becomes a 2nd-order differential equation which has solution given by some special functions. Happily, there is a much better way to solve this problem, which is simpler, more interesting, and more important than solving a differential equation. It uses creation and annihilation operators

$$a \equiv \sqrt{\frac{m\omega}{2\hbar}}(x + ip/m\omega), \qquad \text{so} \qquad a^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}(x - ip/m\omega).$$

These satisfy the fundamental property $[a, a^\dagger] = 1$. We can then immediately show that the Hermitian operator $N = a^\dagger a$ has eigenvalues $n = 0, 1, 2\ldots$, and the eigenvectors $|n\rangle$ satisfy $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n\rangle$. Since $H_{SHO} = \hbar(\omega N + \frac{1}{2})$, we're done. If we really want $\langle x|n\rangle$, we can get it from $|n\rangle = (n!)^{-1/2}(a^\dagger)^n|0\rangle$ by replacing $p \to -i\hbar\frac{d}{dx}$ and we can solve for $\psi_0(x) = \langle x|0\rangle$ by using $a|0\rangle = 0$, which in position space becomes a simple first-order differential equation for $\psi_0(x)$:

$$\psi_n(x) = \frac{(m\omega/2\hbar)^{n/2}}{\sqrt{n!}}(x - \frac{\hbar}{m\omega}\frac{d}{dx})^n\psi_0(x), \qquad (x + \frac{\hbar}{m\omega}\frac{d}{dx})\psi_0(x) = 0.$$

In the Heisenberg picture we have $\dot{a} = -i\omega a$, hence $a(t) = e^{-i\omega t}a$, where $a = a(0)$.

• Path integral description of QM, solenoids and observability of flux inside. Dirac's magnetic monopoles and quantization rule.

• Klein Gordon theory, SHO, and charged Klein Gordon theory. Covariant derivatives and minimal substitution. Euler Lagrangian equations for field theory.

• $S = \int d^4x(-1/4)F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu/c$.

• Dirac equation and Lagrangian